PREFACE

This volume contains the proceedings of the International Conference on Harmonic Analysis and Related Topics held at Macquarie University, Sydney, from January 14–18, 2002. The conference celebrated the many significant achievements and contributions to mathematics of Professor Alan G. R. M^{c}Intosh, on the occasion of his sixtieth birthday. A large number of the speakers were his mathematical protegés and co-authors, and all expressed their gratitude to Alan for the way he had influenced or nurtured their careers.

The conference brought together researchers working in harmonic and functional analysis, linear and non-linear partial differential equations, functional calculus and operator theory, and included researchers from North America, Europe and Asia as well as Australia.

We gratefully acknowledge the support of the contributors to this volume and the sponsors of the conference, the Centre for Mathematics and its Applications at ANU and the Department of Mathematics and the Division of Information and Communication Sciences at Macquarie University. Our thanks also go to Dr Chris Wetherell for his editorial assistance in preparing this volume. We especially thank Alan M^{c}Intosh for his participation. A short synopsis of Alan's career, and a full list of his publications to date are included in this volume.

Contributions for the proceedings were invited from all participants. All papers received were carefully refereed by peer referees.

Xuan Thinh Duong, Alan Pryde
(Editors)

FOREWORD

Alan McIntosh spent his undergraduate years at the University of New England in Armidale NSW where he obtained his BSc(Hons) in 1962. He gained his PhD at the University of California, Berkeley in 1966 under František Wolf. After a year at the Institute for Advanced Study, Princeton, he returned to Australia and began a long association with Macquarie University. In 1999, after 32 years at Macquarie, he became Head of the Centre for Mathematics and its Applications at ANU.

Throughout his time at Macquarie, McIntosh provided leadership in analysis. For many years his group ran weekly seminars which typically attracted participants from the other universities in Sydney. Many well-known mathematicians presented lectures. McIntosh made strong post-doctoral appointments with a diversity of backgrounds and fostered excellence in research. One of his major goals was and remains to nurture young mathematicians. The result has been a long list of outstanding mathematical advances from McIntosh and those who have been fortunate to come under his influence.

McIntosh's early work on accretive bilinear forms was heavily influenced by Tosio Kato (Berkeley), and it was a problem posed by his mentor in 1960 that led to McIntosh's most significant work. The Kato square root problem asks whether the square root of an accretive operator in divergence form is stable under perturbations of the original operator. It was not until 1981 that the one dimensional version of this problem was solved in the fundamental work of Ronald Coifman (Yale), McIntosh and Yves Meyer (ENS-Cachan). In this paper [11], the authors also solved the conjecture of Calderón on the boundedness of the Cauchy integral on a Lipschitz curve. It took until 2000 for the two dimensional version of the problem to fall at the hands of Steve Hofmann (Missouri) and McIntosh. In the following year, the full arbitrary dimensional solution was given in the joint work of Pascal Auscher (Paris-Sud), Hofmann, Michael Lacey (Georgia Tech.), McIntosh and Philippe Tchamitchian (Marseille). Beginning with his collaboration with Coifman and Meyer, McIntosh was to forge remarkable links between the harmonic analysis of the Zygmund school and the operator theory of Kato and others. This work involves the use of square function estimates associated with particular operators, and the construction of the corresponding functional calculi, and depends upon L^p estimates for singular integrals. One specific aim is to study boundary value problems for linear partial differential equations with non-smooth coefficients on irregular domains, and associated nonlinear problems. Results are obtained under natural geometric conditions and these are of special interest when applied to nonlinear problems arising from physical or geometric phenomena. They also have implications for parabolic and hyperbolic problems. The principal methods involve developing the harmonic analysis of operators directly on domains or on their boundaries.

Topics related to M^c^Intosh's research include boundedness of singular integrals and Fourier multipliers on Lipschitz surfaces; heat kernel bounds and functional calculi of elliptic partial differential operators; compensated compactness; spectral theory and functional calculi of operators; and Clifford analysis.

M^c^Intosh has been a Fellow of the Australian Academy of Science since 1986. Very recently he was awarded the Moyal Medal for 2002 for his contributions to mathematics, in particular for his fundamental contributions to harmonic analysis and partial differential equations.

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ON A UNIFORM APPROACH TO SINGULAR INTEGRAL OPERATORS

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Abstract. The essence of an approach to the boundedness of singular integral operators based on several parameterized classes of new conditions one of which includes, in particular, Hörmander condition and its known variations is exposed. Most of the attention is paid to the comparison with known results in the same settings. The feature of dependence from some parameters being integer is revealed. As some of applications, the existence of functional calculus and variants of Littlewood-Paley-type decompositions in its terms without any requirements of smoothness or absolute value bounds of the kernels of the corresponding holomorphic semigroups is shown.

1. Introduction

The main goal of this note is to display the idea of a unified point of view on sufficient conditions formulated in the style of the Hörmander one for the boundedness of singular integral operators and to motivate its usefullness by means of comparison with known close results (including the theory of Calderón-Zygmund operators). In particular, the here introduced $\mathcal{AD}$-classes of singular integral operators extend and generalize Calderón-Zygmund operators and closely related operators possessing H^∞-calculus. In line of this main purpose, formulations of assertions under consideration are partly included also in more general forms. We consider also the definitions of $\mathcal{AD}$-classes in reduced forms while the complete ones are represented in [22]. The same source contains results on boundedness of singular integral operators (SIO) from one smooth function space to another (corresponding to the "upper case" in the sense of Section 3 below) not included in this note too.

The theory of singular integral operators has a half-century background of intensive development. The main ingredient — decomposition lemma — appeared in 1952 thanks to A.P. Calderón and A. Zygmund (see [6]). In 1960, L. Hörmander (see [14])introduced his

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"Cancellation condition", and, since that time, the notion of singular integral operator (SIO) is understood as follows. A SIO T is an integral operator defined, in a sense, by means of the kernel K, s.t. $T \in \mathcal{L}(L_{p_0}, L_{p_0})$:

$$Tf(x) = p.v. \int K(x,y)f(y)dy.$$

The Hörmander condition states that for any $y, z \in \mathbb{R}^n$, and some $C > 0$

$$\int_{|x-z|\geq 2|y-z|} |K(x,y) - K(x,z)|dx \leq C_H < +\infty. \qquad \text{(class } \mathcal{H}\text{)}$$

Every SIO satisfying $(\mathcal{H})$ (from the class $\mathcal{H}$) is bounded (admits an extention) from L_p to $L_p, 1 < p < \infty$, from H_1 to L_1 and from L_1 to $L_{1,\infty}$ (weak-L_1). In addition, the adjoint operator is bounded from L_∞ to BMO.

Presented in this note results with the same statements as just mentioned are discussed in the section 3. One can point out that another condition weaker then $\mathcal{H}$ was presented by X. Duong and A. M[c]Intosh (1999) in [9], and our approach permits to weaken it in the same settings (see $\mathcal{AAD}$-condition in [22]).

Nowadays the following definition of Calderón-Zygmund operator (CZO) is the most commonly accepted.

A CZO is a SIO T satisfying for some $0 < \delta < 1$:

a) $|K(x,y)| < C/|x-y|^n$;

b) $|K(x,y) - K(x,z)| \leq C|y-z|^\delta |x-z|^{-(n+\delta)}$ for $|x-z| \geq 2|y-z|$

We are not imposing absolute value conditions like *a*) at all but one of the introduced here $\mathcal{AD}$-classess contains conditions which are equivalent , or weaker then the above mentioned ones. Namely, $\mathcal{AD}_x(L_1, \infty, 0, 0, 0)$ is equivalent to the Hörmander integral condition, and $\mathcal{AD}_x(L_\infty, \infty, \delta, \delta, \delta)$ in def. 2.5 is weaker then property *b*) of Calderón-Zygmund operator with another one.

In 1972, C.L. Fefferman, E.M. Stein (see [12]) proved (particularly) $H_1 - L_1$ and $L_\infty - BMO$-boundedness of Calderón-Zygmund operators.

Let us pay more attention to the H_p-theory of SIOs.

R. Coifman (see [7]) obtained (1974) $H_p - H_p$ boundedness of CZO for the case of one dimension. In 1986, J. Alvarez and M. Milman (see [3]) established $H_p - H_p$-boundedness excluding the limiting cases (i.e. $p > n/(n+\delta), 0 < \delta < 1$). More precisely their assertion reads as follows: a CZO T satisfying the orthogonality condition $T^\star \mathcal{P}_0 = 0$ is bounded on H_p. Here the orthogonality condition $T^\star \mathcal{P}_N = 0$ means $\int x^\alpha Ta = 0$ for any $|\alpha| \le N$, and $a \in C_0^\infty$ orthogonal to $\mathcal{P}_N$.

Extension of this result to $0 < p < 1$ was pointed out by several authors: δ-CZO satisfying $T^\star \mathcal{P}_{[\delta]}$ is bounded on H_p for $0 < p \le 1$. Next we recall the definition of δ-CZO.

Let $s = 1$ for $\delta \in \mathbb{N}$, and $s = \{\delta\}$ otherwise. Let T be a SIO, then it is δ-CZO if it satisfies

a) $|K(x,y)| < C/|x-y|^n$;

b) $|D_y^\alpha K(x,y) - D_y^\alpha K(x,z)| \le C|y-z|^s|x-z|^{-(n+|\alpha|+s)}$ for $|x-z| \ge 2|y-z|, |\alpha| = [\delta]$.

Similarly to the case of CZOs, some of the presented $\mathcal{AD}$-conditions (for example, $\mathcal{AD}_x(\infty, L_\infty, l_\infty, \delta, \delta, \delta)$) in this note are weaker then condition *b)* of the δ-CZOs. We provide a direct analog of the Hörmander condition in this case too (e.g. $\mathcal{AD}_x(u, L_q, l_\infty, \delta, \delta, \delta), u, q \in [1, \infty)$).

One should add that J. Alvarez (1992) (see [2]) showed the lack of $H_p - L_p$ (and $H_p - H_p$) boundedness for $p = n/(n+\delta)$. In 1994, D. Fan (see [10]), exploiting Littlewood-Paley-theory approach, considered the limiting case for a convolution δ-CZO T, that is, he demonstrated that, under the above conditions, T is bounded from H_p to $H_{p,\infty}$.

R. Fefferman and F. Soria (1987) (see [13]) proved $H_{1,\infty} - L_{1,\infty}$-boundedness for a convolution SIO T satisfying the following Dini condition:

$$\int_0^{1/2} \Gamma(t)dt/t < \infty, \text{ where } \Gamma(t) = \sup_{h \ne 0} \int_{|x|>2|h|/\delta} |K(x-h) - K(x)|dx.$$

In 1988 (publ. 1991), using a similar approach, H. Liu (see [15]) investigated boundedness properties of a convolution SIO (in particular, a CZO) in the setting of homogeneous groups and obtained the following results:

a) $H_p = H_{p,\infty}$-boundedness for CZO (without condition *a)*), if $n/(n+1) < p < 1$;

b) $H_{p,\infty} - L_{p,\infty}$-boundedness, if $n/(n+1) < p \le 1$, for SIO T satisfying

$$\int_0^{1/2} \Gamma(t)^p |\log t| t^{np-1-n} dt < \infty, \text{ where}$$

$$\Gamma(t) = \sup_{h \ne 0} \int_{|h|/\delta < |x| < 4|h|/\delta} |K(x-h) - K(x)|dx;$$

c) $H_{p,\infty} - H_{p,\infty}$-boundedness, if $n/(n+1) < p < 1$, for a ω-CZO (without *a)*-cond.), that is for a SIO T satisfying

$$|K(x-y) - K(x)| < C|x|^{-n}\omega(|y|/|x|), |x| > 2|y|,$$

where ω is a nondecreasing function with

$$\int_0^{1/2} t^{n-n/p-1} |\log t|^{2/p+\varepsilon} \omega(t) dt < +\infty \text{ for some } \varepsilon > 0.$$

But, for $\omega(t) = t^s$, one needs $s > n/p - n$, i.e. a nonlimiting case.

The theorems in the 5th section contain extensions or additions (including the case $0 < p \leq n/(n+1)$) to most of these results concerning the H_p-theory (the additions to the rest and complete proofs are in [22]).

It is interesting to point out the "off-diagonal" case of the Calderón-Zygmund-Hörmander result on boundedness of a SIO proved in 1961 by J.T. Schwartz (see [20] and an extension due to H. Triebel [21]): for $1 < p_0 \leq r_0 < \infty, 1 \leq q$, $1 + 1/r_0 = 1/p_0 + 1/q$, suppose that a convolution operator T with kernel K is bounded from L_p to L_r

$$\text{and satisfies } \int_{|x|>2|y|} |K(x-y) - K(x)|^q dx \leq C.$$

Then T is bounded from L_p to L_r for $1 < p \leq p_0$, $1+1/r = 1/p+1/q$ and from L_1 to $L_{q,\infty}$. But this ("off-diagonal") setting of the SIO theory has not attracted much attention since that time even in spite of the work ([17], 1963) of P.I. Lizorkin on (L_p, L_q)-multipliers theorem. All the general forms of the assertion of this note include the "off-diagonal" case.

Some other positive features of our approach to the boundedness of SIO are the following: $a)$ cases of operators sending Hardy-Lorentz space to both Hardy-Lorentz and Lorentz space or, acting between spaces of smooth functions are covered; $b)$ a new effect of the dependence from some parameters being integer (in particular, the case $n/p \in \mathbb{N}$ (in isotropic situation) for H-space theory) has been observed; $c)$ in some cases parameters of the "target" space are shown to be optimal; $d)$ the case $0 < p \leq n/(n+1)$ for the $H-H$-boundedness does not require smoothness assumptions; $e)$ some obtained sufficient conditions of boundedness of a SIO under consideration have stronger known analogs but the other do not; $f)$ the approach admits consideration an anisotropically SIO in the sense of [5](1966); $g)$ the proofs of the results analogous to classical ones are no more complicated then their counterparts.

The section 4 is devoted to some applications of the considered here and in [22] general forms of SIO boundedness assertions to questions connected with functional calculus. There we consider the class of operators with the kernels of their holomorphic semigroups satisfying Poisson-like $\mathcal{AD}$-estimates introduced in this note and providing, in this terms, sufficient conditions for the existence of a functional calculus of some operators in Hardy and other function spaces, extending results of D. Albrecht, X. Duong and A. M^{c}Intosh (1995) (see [1, 8]). For the limiting values of parameters, the norm estimate (for the bounded extension) of the form

$$\|\phi(T)\|_{\mathcal{L}(X,Y)} \leq C\|\phi\|_{H_\infty}$$

is proved for X being a Hardy space and Y — a Hardy-Marcinkiewicz, so that $X \neq Y$.

Another application is a continuous, in the sense of [19] (1972), form of Littlewood-Paley theorem in term of the above mentioned functional calculus. It should be noted that classical approach to this theorem relied on the properties of Hilbert transforms even in weighted multiple (product) case (see [18]) (1967). Instead, in this note we are following the approach of direct application of vector-valued SIO boundedness results used by O.V. Besov (1984) in [4] to extend Littlewood-Paley inequality to the L_p-spaces with mixed norm of functions periodic in some directions and by X. Duong (1990) to extend the existence of H^∞-functional calculus on L_2 to one in $L_p, p \neq 2$.

The author is pleased to thank O.V. Besov for formulating the general problem to extend classical Hörmander SIO boundedness result to $H_p(L_p)$ spaces with $p < 1$ several years ago, A. M^{c}Intosh and A. Sikora for formulating the problem to obtain a Littlewood-Paley type characterization of Hardy spaces in terms of functional calculus and A. M^{c}Intosh with T. ter Elst for their comments improving the manuscript of this paper. This work was conducted in The Center for Mathematics and its Applications of The Australian National University.

2. Definitions and Designations.

Assume $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set E, $n \in \mathbb{N}$ let E^n be the Cartesian product. Let A be a Banach space and denote by $\|\cdot|A\| = \|\cdot\|_A$ the norm in space A. For $t \in (0,\infty]$ let l_t be a (quasi)normed space of sequences with finite (quasi)norm $\|\{\alpha\}|l_t\| = \left(\sum_i |\alpha_i|^t\right)^{1/t}$ for $t \neq \infty$, or $\|\{\alpha\}|l_\infty\| = \sup_i |\alpha_i|$; Assume also designation $l_{t,log}$ for the (quasi)normed space of sequences with the finite norm $\|\{\alpha\}|l_{t,log}\| = \|\{\beta\}|l_t\|$, where $\beta_j = \sum_{i\geq j} |\alpha_i|$. For an measurable subset G of $\mathbb{R}^n$, let $X(G,A)$ be a function space of all (strongly) measurable functions $f : G \to A$ with some quasiseminorm $\|\cdot|X(G,A)\|$. In particular, for $p, q \in (0,\infty]$ let $L_{p,q}(G,A)$ be the Bochner-Lebesgue-Lorentz space of all (strongly) measurable functions $f : G \to A$ with the finite norm $\|f|L_{p,q}(G, A)\| = \|\|f\|_A|L_{p,q}(G)\|$.

Let $Q_0 := [-1, 1]^n$, $Q_t(z) := z + tQ_0$ for $t > 0, z \in \mathbb{R}^n$. Let $\mathcal{P}_\lambda(A)$ be the space of polynomials $\{\sum_{|\alpha|\le\lambda} c_\alpha x^\alpha : c_\alpha \in A\}$.

Definition 2.1. For $u \in [1,\infty], t > 0, \lambda \ge 0, x \in \mathbb{R}^n, f \in L_{1,loc}(\mathbb{R}^n, A)$, we shall refer to the following local approximation functional by means of polynomials as to the $\mathcal{D}-$ functional:

$$\mathcal{D}_u(t, x, f, \lambda, A) = t^{-n/u}\|f - P_{t,x,\lambda}f|L_u(Q_t(x), A)\|,$$

where $P_{t,x,\lambda} : L_u(Q_t(x)) \to \mathcal{P}_\lambda(A)$ is a surjective projector. For simplicity, we shall understood $\mathcal{D}_u(t, x, f, \lambda)$ to be $\mathcal{D}_u(t, x, f, \lambda, A)$, if $A = \mathbb{R}, \mathbb{C}$. If function f depends also from the two (vector) variables x, y , $f = f(x, y)$, and $f_{|x=w}(y) := f(w, y)$ then

$$\mathcal{D}^y_u(t, z, f(w, \cdot), \lambda, A) = \mathcal{D}_u(t, z, f_{|x=w}, \lambda, A).$$

Let $C_0^\infty(G)$ be the space of infinitely differentiable and compactly supported in the open set G functions.

Define the local maximal functional on a function f by

$$M(t, x, f) := \sup\{|t^{-n}\varphi(\cdot/t) * f|(y) : |y - x| \le \varpi t, \varphi \in C_0^\infty(Q_0)\}.$$

Definition 2.2. For $p, q \in (0,\infty]$, let $H_{p,q}(\mathbb{R}^n, A)$ be a completion of quasinormed space of locally summable A-valued functions f with a finite quasinorm

$$\|f|H_{p,q}(\mathbb{R}^n, A)\| := \left\|\sup_{t>0} M(t, \cdot, f)\,|L_{p,q}(\mathbb{R}^n)\right\|.$$

Remark 2.3. It will be used that $H_{p,q}(\mathbb{R}^n, A) = L_{p,q}(\mathbb{R}^n, A)$ for $p > 1$.

Throughout the article we shall deal with particular cases of the following one. For $\varphi \in C_0^\infty$, $b > 1$, $\chi_{Q_0} \le \varphi \le \chi_{bQ_0}$, let

$$(Tf)(x) := \int K(x, y)f(y)dy := \lim_{\varepsilon\to 0}\int K(x, y)f(y)\varphi(\varepsilon(y - x))dy,$$

where be a singular integral operator. Moreover, the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$ is measurable and such that for almost every $x \in \mathbb{R}^n$ the function $K(x, \cdot) \in L_1^{loc}(\mathbb{R}^n \setminus \{x\})$. We shall also assume that the operator T is bounded from L_{θ_0} into L_{θ_1} for some $\theta_0, \theta_1 \in (0,\infty)$, and to be in the union of the following classes.

Remark 2.4. More general case of operator-valued kernels corresponding operators T defined on vector-valued functions in the settings, particularly, of the section 5 is considered in [22] and used in the section (4) .

Definition 2.5. Assume $\lambda_0, \lambda_1, \gamma \in [-n, \infty)$, $u, q, q_1 \in (0, \infty]$, $\gamma \geq 0$, $\delta > 0, b > 1$. Let $X := X(\mathbb{N}_0)$ be a (quasi)(semi)normed space of sequences, $E^w_{q,q_1,\lambda_1}(\mathbb{R}^n)$ be the weighted Lorentz space with the norm

$$\|f|E^w_{q,q_1\lambda_1}(\mathbb{R}^n)\| = \left\| f(\cdot)| \cdot -w|^{\lambda_1+n/q'} \mid L_{q,q_1}(\mathbb{R}^n)\right\|,$$

and $\Delta_i(r,w) = Q_{\delta r b^{i+1}}(w) \backslash Q_{\delta r b^i}(w), i \in \mathbb{N}$. Then it will be understood that:

a) $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, \gamma)$, or

b) $T \in \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, \gamma)$, if, correspondingly, the sequence

$$\mu_i(r,w) := \|r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\Delta_i}(\cdot), \gamma)|E^w_{q,q,\lambda_1}(\mathbb{R}^n)\|, i \in \mathbb{N}_0, \text{ or}$$

$$\mu_i(r,w) := r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\Delta_i}(\cdot), \gamma, E^w_{q,q,\lambda_1}(\mathbb{R}^n)), i \in \mathbb{N}_0,$$

is bounded in X by a constant $C > 0$ uniformly in $r > 0, w \in \mathbb{R}^n$;

c) $T \in \mathcal{AD}_x(u, L_{q,q_1}, \lambda_0, \lambda_1, \gamma)$, or

d) $T \in \mathcal{AD}_x(L_{q,q_1}, u, \lambda_0, \lambda_1, \gamma)$, if, correspondingly, the function

$$\mu(r,w) := \|r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)| \cdot -w|^{\lambda_1+n/q'}, \gamma)|L_{q,q_1}(\mathbb{R}^n \setminus Q_{r\delta}(w))\|, \text{ or}$$

$$\mu(r,w) := r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\mathbb{R}^n \setminus Q_{r\delta}(w)}(\cdot), \gamma, E^w_{q,q_1\lambda_1}(\mathbb{R}^n)),$$

is bounded by a constant $C > 0$ uniformly in $r > 0, w \in \mathbb{R}^n$.

The infimum of constants C in each case will be designated by means of $C_{\mathcal{AD}}$ for the corresponding $\mathcal{AD}$-condition.

Remark 2.6. It can be noted that the definitions of $\mathcal{AD}$-classes have and equivalent continuous forms, which means also their independence from the parameter $b > 1$.

Definition 2.7. Let $\gamma_0, \gamma_1 \geq 0$. An operator T will be assumed form the class $ORT_x(\gamma_0, \gamma_1)$ if $\int \pi T \phi = 0$ for all $\pi \in \mathcal{P}_{\gamma_1}$ and $\phi \in C^\infty_0$, such that $\int \phi \pi = 0$ for each $\pi \in \mathcal{P}_{\gamma_0}$.

Definition 2.8. For $\Omega \subset \mathbb{C}$ let $\{T(z)\}_{z\in\Omega}$ be a family of integral operators with the corresponding $\mathbb{C}$-valued kernels $\{K_z\}_{z\in\Omega}$, $K_z = K_z(x,y)$, $x, y \in \mathbb{R}^n$. We shall assume that the family $\{T(z)\}_{z\in\Omega}$ satisfies Poisson-type $\mathcal{AD}_x$-estimates with parameters $u \in [1,\infty], \lambda \geq 0$ on the domain Ω if for some $\epsilon, m \in (0, \infty)$ and any $w, x \in \mathbb{R}^n, z \in \Omega, r \in (0, \infty)$

$$\mathcal{D}_u(r, w, K_z(x, \cdot), \lambda) \leq C\left(\frac{r}{|z|^m}\right)^\lambda |z|^{-mn}\left(1 + \frac{|x-w|}{|z|^m}\right)^{-(n+\lambda+\epsilon)},$$

$T(z) \in ORT_x(\lambda, \lambda)$.

And $K_z(x,y)$ is understood satisfying Poisson-type $\mathcal{AD}_y$-estimate if $K^I_z(x,y) = K_z(y,x)$ satisfies Poisson-type $\mathcal{AD}_x$-estimate.

Definition 2.9. We shall understood operator T defined by the kernel $K(x,y)$ to be in $\mathcal{AD}_y$-class, or $ORT_y(\gamma_0,\gamma_1)$-class if, and only if, the corresponding operator T^I defined by the kernel $K^I(x,y) = K(y,x)$ is in the corresponding $\mathcal{AD}_x$-class, or, correspondingly, $ORT_x(\gamma_0,\gamma_1)$-class.

3. Counterparts of Known (Clasical) Results.

For the sake of simplicity, we shall only consider in this section $\mathcal{AD}$-classes with $q = 1$, $X = l_1$ and $\lambda_0 = \lambda_1 = \gamma = 0$.

Remark 3.1. In spite of the relation

$$\mathcal{AD}_x(u, L_1, 0, 0, 0) = \mathcal{AD}_x(u, L_1, l_1, 0, 0, 0) \subset$$

$$\subset \mathcal{AD}_x(L_1, u, l_1, 0, 0, 0) \subset \mathcal{AD}_x(L_1, u, 0, 0, 0),$$

not coinciding $\mathcal{AD}$-classes will be discussed in the proofs separately to demonstrate the approach in more general cases.

Let us point out that the class of operators satisfying Hörmander condition $\mathcal{H}$ is equal to the class $\mathcal{AD}_x(L_1, \infty, l_1, 0, 0, 0)$. We shall show the inclusion $\mathcal{H} \subset \mathcal{AD}_x(L_1, \infty, l_1, 0, 0, 0)$. The opposite one was pointed out to the author by A. M^c^Intosh. Indeed, the corresponding kernels should have a uniformly bounded for any $z \in \mathbb{R}^n$, $r > 0$ quantity

$$A(r,z) = \inf_{c(x)} \sup_{\{y : |y-z| \le r\}} \int_{|x-z| \ge 2r} |K(x,y) - c(x)| dx.$$

And, supposing, for fixed z, r, $c(x)$ to be equal to $K(x,z)$, we can note that

$$A(r,z) \le \sup_{\{y : |y-z| \le r\}} \int_{|x-z| \ge 2r} |K(x,y) - K(x,z)| dx \le$$

$$\le \sup_y \int_{|x-z| \ge 2|y-z|} |K(x,y) - K(x,z)| dx \le C_H < +\infty,$$

where C_H is the constant in Hörmander condition (class $\mathcal{H}$).

3.1. **Lower "Summability" Case.**

Theorem 3.2. *For $p_0 \in (1, \infty]$, $u \in [1, \infty]$, let T be a SIO from $\mathcal{AD}_x(L_1, u, l_1, 0, 0, 0) \cup \mathcal{AD}_x(L_1, u, 0, 0, 0) \cup \mathcal{AD}_x(u, L_1, 0, 0, 0)$, bounded from L_{p_0} into itself. Then,*

a) $T \in \mathcal{L}(L_{p,q}(\mathbb{R}^n))$ *for* $p \in (1, p_0]$, $q \in (0, \infty]$;
b) $T \in \mathcal{L}(L_1(\mathbb{R}^n), L_{1,\infty}(\mathbb{R}^n))$ *if* $u = \infty$;
c) $T \in \mathcal{L}(H_1(\mathbb{R}^n), L_1(\mathbb{R}^n))$.

Proof of the Theorem 3.2. We suppose that kernel $K(x,y)$ corresponds to the operator T. One should note that part $a)$ of the theorem is a consequence of both $b)$ and $c)$ in view of the interpolation properties of the scale of Hardy-Lebesgue spaces (see [11]). To the first, let us recall that the statements of the parts $b)$ and $c)$ are implied as in classical approach) by the estimate

$$\int_{\mathbb{R}^n \setminus Q_{r\delta}(z)} |Ta| dx \leq C_{\mathcal{AD}}, \tag{1}$$

where $r > 0, z \in \mathbb{R}^n$ and a is a $(1, L_\infty, 0)$-, or a $(1,1,0)$-atom in the case of the part $c)$, or $b)$ correspondingly. Indeed, in the case $c)$, the atomic decomposition result for H_1 (see [7, 16]) permits us to prove the boundedness of T on $(1,\infty,0)$-atoms only, what follows from (1) and

$$\int_{Q_{r\delta}(z)} |Ta| dx \leq (r\delta)^{n/p_0'} \|Ta\|_{p_0} \leq \delta^{n/p_0'} \|T|\mathcal{L}(L_{p_0})\|,$$

where a is an $(1,\infty,0)$-atom. In the case $b)$, for a function $f \in L_1$ and $\lambda > 0$, Calderón-Zygmund decomposition of a set $\Omega_\lambda := \{x : Mf > \lambda\} = \bigcup_{i\in\mathbb{N}} Q_i$, where the set $\{\delta Q_i\}$ possess finite intersection property and $C|\Omega_\lambda| \geq \sum_i |Q_i|$, provides representation

$$f = f_0 + C\lambda \sum_i |Q_i| a_i, \text{ where } a_i \text{ is a } (1,1,0) - \text{atom} \tag{2}$$

and $\|f_0|L_\infty\| \leq C\lambda$. Therefore, Chebyshev inequality, (1) and just mentioned properties imply

$$\lambda\{|Tf_0| > c\lambda\} \leq C\lambda^{1-p_0} \|Tf_0\|_{p_0}^{p_0} \leq C \|T|\mathcal{L}(L_{p_0})\|^{p_0} \|f|L_1\|,$$

$$\lambda\{|Tf_1| > c\lambda\} \leq C\lambda \left(|\cup_i \delta Q_i| + \sum |Q_i| \int_{\mathbb{R}^n \setminus \delta Q_i} |Ta_i| \right) \leq$$

$$\leq C\lambda |\Omega_\lambda| \leq C \|f|L_1\|. \tag{3}$$

To obtain the formula (1) suppose $g(x)$ to be a function minimizing functionals

$$\inf_c \int_{Q_r(z)} |K(x,y) - c|^u dy \tag{4}$$

at a.e. x if $T \in \mathcal{AD}_x(u, L_1, l_1, 0, 0, 0)$, or minimizing the functional

$$\int_{Q_r(z)} \left(\int_{\mathbb{R}^n \setminus Q_{r\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy, \tag{5}$$

if $T \in \mathcal{AD}_x(L_{1,1}, u, 0, 0, 0)$, or $g(x) = \sum_i g_i(x)$, where functions $\{g_i(x)\}_{i\in\mathbb{N}}$ to minimize functionals

$$\inf_{c(x)} \left(\int_{Q_r(z)} \left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy \right)^{1/u} \tag{6}$$

correspondingly if $T \in \mathcal{AD}_y(L_1, u, l_1, 0, 0, 0)$. In view of Minkowski inequality, it follows from (4), (5), or (6) that, correspondingly, for an arbitrarily $(1,1,0)$-(part *b*)), or $(1,\infty,0)$-atom with support $Q_r(z)$, one has due to the Hölder and Minkowski inequalities, Fubini theorem and the orthogonality of atom a to constants:

$$\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |Ta| dx \leq$$

$$\leq Q = \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} \left| \int (K(x,y) - g(x)) a(y) dy \right| dx \leq$$

$$\leq \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} r^{-n/u} \left(\inf_c \int_{Q_r(z)} |K(x,y) - c|^u dy \right)^{1/u} dx \leq C_{\mathcal{AD}}, \tag{7}$$

$$Q \leq \int_{Q_r(z)} \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y) - g(x)| dx |a(y)| dy \leq$$

$$\leq \inf_c r^{-n/u} \left(\int_{Q_r(z)} \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy \right)^{1/u} \leq C_{\mathcal{AD}}, \tag{8}$$

$$Q \leq \sum_{i\in\mathbb{N}} \int_{Q_r(z)} \int_{Q_{2^i r\delta}(z)\setminus Q_{2^{(i-1)} r\delta}(z)} |K(x,y) - g_i(x)| dx |a(y)| dy \leq$$

$$\leq \sum_{i\in\mathbb{N}} \inf_c r^{-n/u} \left(\int_{Q_r(z)} \left(\int_{Q_{2^i r\delta}(z)\setminus Q_{2^{(i-1)} r\delta}(z)} |K(x,y) - c(x)| dx \right)^u dy \right)^{1/u} \leq C_{\mathcal{AD}}. \tag{9}$$

In this manner, estimates $(7-9)$ motivate (1).

3.2. **Upper "Summability" Case.** The next theorem can be derived from the previous one by means of duality considerations but such approach will not work definitely, in particular, in the case of vector-valued functions, or will require additional duality results to consider scales other than $H_1 - L_p - BMO$. Thus, proof provided does not rely on duality.

Theorem 3.3. *For* $p_0 \in (1,\infty]$, $u \in [1,\infty)$, *let* T *be a SIO from* $\mathcal{AD}_y(L_1, u, l_1, 0, 0, 0) \cup \mathcal{AD}_y(L_1, u, 0, 0, 0) \cup \mathcal{AD}_y(u, L_1, 0, 0, 0)$, *bounded from* L_{p_0} *into itself. Then,*
a) $T \in \mathcal{L}(L_{p,q}(\mathbb{R}^n))$ *for* $p \in [p_0, \infty)$, $q \in (0,\infty]$;
b) $T \in \mathcal{L}(L_\infty(\mathbb{R}^n), BMO(\mathbb{R}^n))$.

Proof of the Theorem 3.3. We suppose that kernel $K(x,y)$ corresponds to the operator T. One should note that we need to prove the part $b)$ only because the part $a)$ follows from it with the aid of the real interpolation method.

Let us fix $Q_r(z) \subset \mathbb{R}^n$, $f \in L_\infty$ and use representation $f = f_0 + f_1$, $f_0 = \chi_{Q_{r\delta}(z)}$, where δ is a constant in the definition of the corresponding $\mathcal{AD}_y$-classes. Then the definition of $\mathcal{D}$-functional and L_{p_0}-boundedness of T and restriction operator $f \longrightarrow f_0$ imply

$$\mathcal{D}_{p_0}(r,z,Tf_0,0) \le r^{-n/p_0}\|Tf_0|L_{p_0}(\mathbb{R}^n)\| \le r^{-n/p_0}\|T|\mathcal{L}(L_{p_0})\|\times$$
$$\times\|f_0|L_{p_0}(\mathbb{R}^n)\| \le \|T|\mathcal{L}(L_{p_0})\|\|f_0|L_\infty(\mathbb{R}^n)\| \le \|T|\mathcal{L}(L_{p_0})\|\|f|L_\infty(\mathbb{R}^n)\|. \tag{1}$$

Now suppose $g(y)$ to be a function minimizing functionals

$$\inf_c \int_{Q_r(z)} |K(x,y)-c|^u dx \tag{2}$$

at a.e. y if $T \in \mathcal{AD}_y(u, L_1, l_1, 0, 0, 0)$, or minimizing the functional

$$\int_{Q_r(z)} \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y)-c(y)|dy \right)^u dx, \tag{3}$$

if $T \in \mathcal{AD}_y(L_{1,1}, u, 0, 0, 0)$, or $g(y) = \sum_i g_i(y)$, where the functions $\{g_i(y)\}_{i\in\mathbb{N}}$ to minimize functionals

$$\inf_{c(y)} \left(\int_{Q_r(z)} \left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)} |K(x,y)-c(y)|dy \right)^u dx \right)^{1/u} \tag{4}$$

correspondingly if $T \in \mathcal{AD}_y(L_1, u, l_1, 0, 0, 0)$. In view of Minkowski inequality, it follows from (2), or (3), or (4) that, correspondingly,

$$\mathcal{D}_u(r,z,Tf_1,0) \le r^{-n/u} \left(\int \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |(K(x,y)-g(y))f_1(y)|dy) \right)^u dx \right)^{1/u} =$$

$$= Q \le \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} \left(\int_{Q_r(z)} |K(x,y)-g(y)|^u dx \right)^{1/u} \|f|L_\infty dy, \text{ or} \tag{5}$$

$$Q \le \left(\int \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y)-g(y)|dy \right)^u dx \right)^{1/u}, \text{ or} \tag{6}$$

$$Q \le \left(\sum_i \left(\int_{Q_r(z)} \left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)} |K(x,y)-c(y)|dy \right)^u dx \right)^{1/u} \right) \|f|L_\infty\|. \tag{7}$$

Eventually formulas (1) and one of $(5,6,7)$ imply

$$\mathcal{D}_1(r,z,Tf,0) \le (C_{\mathcal{AD}} + \|T|\mathcal{L}(L_{p_0})\|)\|f|L_\infty\|.$$

4. Applications

4.1. **Functional Calculus and Littlewood-Paley-type Theorems.**

In this section we have $A = B = \mathbb{C}$. All the definitions and notations regarding functional calculus are understood as in the article [1] due to David Albrecht, Xuan Duong and Alan M^{c}Intosh, including the following definitions.

For $0 \le \omega < \mu < \pi$ let $S_{\omega+} := \{z \in \mathbb{C} | \, |\arg z| \le \omega\} \cup \{0\}$, $S^0_{\mu+} := \{z \in \mathbb{C} | \, |\arg z| < \mu\}$, space $H(S^0_{\mu+})$ be the space of all holomorphic functions on $S^0_{\mu+}$ endowed with the $L_\infty(S^0_{\mu+})$-norm, containing subspace $\Psi(S^0_{\mu+}) := \{\psi | \psi \in H(S^0_{\mu+}), \exists s > 0, |\psi(z)| \le C|z|^s(1+|z|^{2s})^{-1}\}$. A closed in $L_2(\mathbb{R}^n)$ operator T is said to be of type $S_{\omega+}$ if $\sigma(T) \subset S_{\omega+}$ and for any $\mu > \omega$ there exist C_μ such that

$$|z| \|(T - zI)^{-1} | \mathcal{L}(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))\| \le C_\mu, z \notin S_{\omega+}.$$

Theorem 4.1. *Assume $q \in (0,\infty]$, $r \in (0,\infty]^n$. Let T be a one-one operator of type $S_{\omega+}$ in $L_2(\mathbb{R}^n)$, $\omega \in [0, \pi/2)$, having a bounded functional calculus in $L_2(\mathbb{R}^n)$ for all $f \in H_\infty(S^0_{\mu+})$ for some $\mu > \omega$. Assume that for some $\lambda, m, \epsilon > 0$, $[v_0, \mu] \in (\omega, \pi/2)$ and all $z \in S^0_{(\pi/2-\mu)}$, the kernel $K_z(x,y)$ of holomorphic semigroup e^{-zT} associated with T satisfies:*

a) Poisson-type $\mathcal{AD}_x$-estimate with $u = 1$;

b) Poisson-type $\mathcal{AD}_y$-estimate with $u = 1$;

c) both Poisson-type $\mathcal{AD}_x$- and $\mathcal{AD}_y$-estimates with $u = 2$.

Then, correspondingly, for $f \in H_\infty(S^0_\nu)$ for all $\nu > \mu$, $f(T)$ can be extended to be in:

a) $\displaystyle\bigcup_{p \in ((1+\lambda/n)^{-1}, 2]} \mathcal{L}(H_{p,q}(\mathbb{R}^n), H_{p,q}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(H_{p,q}, H_{p,q})\| \le C\|f|L_\infty\|$ *and*

$\mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,\infty}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,\infty}(\mathbb{R}^n))\| \le C\|f|L_\infty\|$ *for* $p_0 = (1+\lambda/n)^{-1}$, $\lambda \notin \mathbb{Z}$;

b) $\displaystyle\bigcup_{p \in [2,\infty)} \mathcal{L}(L_{p,q}(\mathbb{R}^n), L_{p,q}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(L_{p,q}, L_{p,q})\| \le C\|f|L_\infty\|$

and $\displaystyle\bigcup_{\gamma \in [0,\lambda), T \in \mathcal{ORT}_y(\gamma,\gamma)} \mathcal{L}(X^\gamma, X^\gamma)$ *with* $\|f(T)|\mathcal{L}(X^\gamma, X^\gamma)\| \le C\|f|L_\infty\|$;

for $X^\gamma \in \left\{b^\gamma_{r,q}(\mathbb{R}^n), l^\gamma_{r,q}(\mathbb{R}^n)\right\}$;

c) if, in addition, functional $\Psi_{\nu_0}(f) = \int_{\Gamma_{\nu_0}} \frac{|f(\zeta)|}{|\zeta|} |d\zeta|$ *is finite for some* $\Gamma_{\nu_0} = \Theta(t) t e^{(\pi-\nu_0)/2} + (\Theta(t) - 1) t e^{(\nu_0-\pi)/2}, t \in \mathbb{R}$, *then the following Littlewood-Paley-type estimates is true: for* $\gamma \in [0,\lambda)$, $T \in$

$\mathcal{ORT}_y(\gamma,\gamma)$, $p\in((1+\lambda/n)^{-1},\infty)$ *and*

$$X(\mathbb{R}^n)\in\left\{H_p(\mathbb{R}^n),b^{\gamma}_{\infty,\infty}(\mathbb{R}^n),b^{\gamma}_{r,q}(\mathbb{R}^n),l^{\gamma}_{r,q}(\mathbb{R}^n)\right\}$$

$$\left\| f(tT)g|\,X(\mathbb{R}^n,L_{2,\frac{dt}{t}}(\mathbb{R}_+))\right\| \asymp (\|f|L_\infty\|+\Psi_{\nu_0}(f))\,\|g\mid X(\mathbb{R}^n)\|\,.$$

Partial proof of the Theorem 4.1.We shall discuss only the proofs of the assertions concerning Hardy-Lorentz spaces (”Lower case”). Full proof contained in [22].

Theorem D from [1] supplies an opportunity to consider only functions f from the class $\Psi(S^0_{\mu+})$ in all the parts of the theorem thanks to some limiting procedure.

By the theorem 5.1 and the existence of H^∞-calculus of operator T in L_2, it is sufficient to estimate only the constants $C_{\mathcal{AD}}$ from the definitions of the appropriate $\mathcal{AD}$-classes. For this purpose we shall use the following representation of the operator $f(T)$ and its kernel, obtained by Xuan Duong (see [8]):

$$f(T)=\int_{\Gamma_{\nu_0}}e^{-zT}n(z)dz,\quad n(z)=\int_{\Gamma_0}e^{z\zeta}f(\zeta)d\zeta, \tag{1}$$

$$\Gamma_{\nu_0}=\Theta(t)te^{(\pi-\nu_0)/2}+(\Theta(t)-1)te^{(\nu_0-\pi)/2},\Gamma_0=\Theta(t)te^{\nu_0}+(\Theta(t)-1)te^{-\nu_0},$$

$$t\in\mathbb{R},K_f(x,y)=\int_{\Gamma_0}k_z(x,y)n(z)dz,\quad |n(z)|\le c|z|^{-1}\|f\|_\infty.$$

Now using subadditivity of $\mathcal{D}$-functional we have in the case $a)$ for a function f with $\|f|L_\infty\|\le 1$

$$\mathcal{D}_u(r,w,K_f(x,\cdot),\lambda)\le C\int_{\Gamma_{\nu_0}}\left(\frac{r}{|z|^m}\right)^{\lambda}|z|^{-mn}\left(1+\frac{|x-w|}{|z|^m}\right)^{-(n+\lambda+\epsilon)}|dz|/|z|\le$$

$$\le Cr^\lambda|x-w|^{-(n+\lambda)}. \tag{2}$$

Hence $f(T)\in\mathcal{AD}_x(1,L_\infty,l_\infty,\lambda,\lambda,\lambda)$ and

$$\mathcal{AD}_x(u,L_\infty,l_{t,log},\gamma,\gamma,\lambda),\ \gamma\in[0,\lambda),t\in(0,\infty].$$

uniformly by f. It means the desirable estimate $C_{\mathcal{AD}}(f)\le C\|f|L_\infty\|$. It is left to apply the part $c)$ of the theorem 5.1.

To prove part $c)$ let us fix a (nonzero) function f $\|f|H_\infty\|+\Psi_{\nu_0}(f)\le 1$, satisfying the conditions of $c)$, and such that for $z\in\Gamma_0$

$$\int_0^\infty f^2(tz)\frac{dt}{t}=c_I>0. \tag{3}$$

Now we can define operators $\Lambda: g(x)\to\{(f(tT)g)(x)\}_{t\in\mathbb{R}_+}$,

$$\Lambda^{-1}:\{h(t,x)\}_{t\in\mathbb{R}_+}\to C_I^{-1}\int_{t\in\mathbb{R}_+}f(tT)h(t,x)\frac{dt}{t},$$

which define an isomorphism between $L_2(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n, L_{2,\frac{dt}{t}}(\mathbb{R}_+))$ because of the theorem F from [1]. Hence, analogously to the derivation of the formula (2), subadditivity of the $\mathcal{D}$-functional, Minkowski inequality and finiteness of $\Psi_{\nu_0}(f)$ imply both for $k_f = K_f$ and for $k_f = K_f^I$

$$\mathcal{D}_u(r, w, k_f(x, \cdot), \lambda, L_{2,\frac{dt}{t}}(\mathbb{R}_+)) \le C \int_{\Gamma_{\nu_0}} \left\| \left(r|z \cdot |^{-m}\right)^{\lambda} |z \cdot |^{-mn} \times \right.$$

$$\left. \times \left(1 + |x - w||z \cdot |^{-m}\right)^{-(n+\lambda+\epsilon)} \right| L_{2,\frac{dt}{t}} \Big\| \, |n(z)| \cdot |dz| \le$$

$$\le Cr^{\lambda} |x - w|^{-(n+\lambda)}. \tag{4}$$

It means that
$\Lambda, \Lambda^{-1} \in \mathcal{AD}_x(1, L_\infty, l_\infty, \lambda, \lambda, \lambda) \cap \mathcal{AD}_y(1, L_\infty, l_\infty, \lambda, \lambda, \lambda)$. Thus the proof of the part $c)$ is finished exactly as ones of the parts $a), b)$.

5. Examples of the Formulations of Main Results in a More General Form.

In this section we shall present two theorems which are, in turn, simplified extracts from the first two main theorems in [22].

Theorem 5.1. *Let T be a singular integral operator with kernel $K(x, y)$ satisfying condition $T \in ORT_x(\gamma, \lambda_1)$, $\lambda_1 \in [-n, \infty)$, $\lambda_0, \gamma \ge 0$ and bounded from $L_{\theta_0}(\mathbb{R}^n)$ into $L_{\theta_1}(\mathbb{R}^n)$, $t, \theta_0 \in (0, \infty]$, $\theta_1 \in [1, \infty]$; let also $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, $p_i = (1 + \lambda_i/n)^{-1}, i = 0, 1$, $u, q \in [1, \infty]$, $q_0, q_1 \in (0, \infty]$, $q > p_1$, $\theta_0 > p_0$, and $K(x, y)$ satisfies condition $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, \gamma) \cup \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, \gamma)$. Then for $1/v_1 - 1/v_0 = 1/\theta_1 - 1/\theta_0$ operator T is also bounded with its norm bounded from above by $C(C_{\mathcal{AD}} + \|T|\mathcal{L}(L_{\theta_0}, L_{\theta_1})\|)$ in the following situations. Assuming $\min\{1, t, p_1\} \ge q_0 \ge p_0$ and the space $X = l_{t,log}$ for $\lambda_1 \in \mathbb{Z}$, or $X = l_t$ for $\lambda_1 \notin \mathbb{Z}$*
a) $T \in \mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n), H_{p_1,t}(\mathbb{R}^n)) \cap \mathcal{L}(H_{v_0,s}(\mathbb{R}^n), H_{v_1,s}(\mathbb{R}^n))$ for $v_0 \in (p_0, \theta_0), s \in (0, \infty]$;
b) $T \in \mathcal{L}(H_{v_0,w_0}(\mathbb{R}^n), H_{v_1,s}(\mathbb{R}^n))$ for $v_1 < q, s \in [v_0, \infty]$, $v_1 \le w_0 \le \min(v_1, s, 1)$, and either for $\theta_0 > 1, v_0 \in (p_0, 1]$, or $\theta_0 \le 1, v_0 \in (p_0, \theta_0)$;
c) in particular, for $\lambda_0 = \lambda_1, \theta_0 = \theta_1 > 1$, $s \in (0, \infty]$
$T \in \mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,t}(\mathbb{R}^n)) \bigcap_{p \in (p_0, 1]} \mathcal{L}(H_{p,s}(\mathbb{R}^n), H_{p,s}(\mathbb{R}^n)) \bigcap$
$\bigcap_{p \in (1, \theta_0]} \mathcal{L}(L_{p,s}(\mathbb{R}^n), L_{p,s}\mathbb{R}^n)$.

Theorem 5.2. *Let T be a singular integral operator with kernel $K(x, y)$ bounded from $L_{\theta_0}(\mathbb{R}^n)$ into $L_{\theta_1}(\mathbb{R}^n)$, $t, \theta_0, \theta_1 \in (0, \infty]$; let also $\lambda_0 \ge 0$, $\lambda_1 \in [-n, \infty)$, $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, finite $\gamma \ge 0$, $p_i =$*

$(1+\lambda_i/n)^{-1}, i=0,1$, $u,q\in[1,\infty]$, $q\geq p_1$, $\theta_0>p_0$. *Then for* $1/v_1-1/v_0=1/\theta_1-1/\theta_0$ *operator* $T\in\mathcal{AD}_x(u,L_{q,t},\lambda_0,\lambda_1,\gamma)\cup\mathcal{AD}_x(L_{q,t},u,\lambda_0,\lambda_1,\gamma)$: *is also bounded with its norm bounded from above by* $C(C_{\mathcal{AD}}+\|T|\mathcal{L}(L_{\theta_0},L_{\theta_1})\|)$ *in the following situations. Assuming* $\min\{1,t,p_1\}\geq q_0\geq p_0$ *and* $K(x,y)$ *satisfying condition*
a) $T\in\mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n),L_{p_1,t}(\mathbb{R}^n))\cap\mathcal{L}(H_{v_0,s}(\mathbb{R}^n),L_{v_1,s}(\mathbb{R}^n))$ *for* $v_0\in(p_0,\theta_0), s\in(0,\infty]$;
b) $T\in\mathcal{L}(H_{v_0,w_0}(\mathbb{R}^n),L_{v_1,s}(\mathbb{R}^n))$ *for* $v_1\leq q, s\in[v_0,\infty]$, $v_0\leq w_0\leq\min(v_1,s,1)$, *and either for* $\theta_0>1, v_0\in(p_0,1]$, *or* $\theta_0\leq 1, v_0\in(p_0,\theta_0)$;
c) in particular, for $\lambda_0=\lambda_1,\theta_0=\theta_1$, $s\in(0,\infty]$
$T\in\mathcal{L}(H_{p_0}(\mathbb{R}^n),L_{p_0,t}(\mathbb{R}^n))\bigcap_{p\in(p_0,\theta_0]}\mathcal{L}(H_{p,s}(\mathbb{R}^n),L_{p,s}(\mathbb{R}^n))$.
d) Assuming $u=\infty$ *and either* $v_1=p_1>1$, *or* $v_1=p_1=t=1$, *or under the conditions of part b) with ,* $\gamma=0$, $v_0=1$,

$$T\in\mathcal{L}(L_1(\mathbb{R}^n,A),L_{v_1,\infty}(\mathbb{R}^n,B)),\ \textit{where } 1-1/v_1=1/\theta_0-1/\theta_1.$$

Remark 5.3. It is proved in [22] that the limiting cases of the theorems 5.1, 5.2 cannot be improved in the sense of reducing the value of the parameter t of the "target" space $H_{p_1,t}$ provide $n/p_1\notin\mathbb{N}$, or the space $L(p_1,t)$, correspondingly.

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ALMOST PERIODIC BEHAVIOUR OF UNBOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. A key result in describing the asymptotic behaviour of bounded solutions of differential equations is the classical result of Bohl-Bohr: If $\phi : \mathbb{R} \to \mathbb{C}$ is almost periodic and $P\phi(t) = \int_0^t \phi(s)\,ds$ is bounded then $P\phi$ is almost periodic too. In this paper we reveal a new property of almost periodic functions: If $\psi(t) = t^N\phi(t)$ where ϕ is almost periodic and $P\psi(t)/(1 + |t|)^N$ is bounded then $P\phi$ is bounded and hence almost periodic. As a consequence of this result and a theorem of Kadets, we obtain results on the almost periodicity of the primitive of Banach space valued almost periodic functions. This allows us to resolve the asymptotic behaviour of unbounded solutions of differential equations of the form $\sum_{j=0}^m b_j u^{(j)}(t) = t^N\phi(t)$. The results are new even for scalar valued functions. The techniques include the use of reduced Beurling spectra and ergodicity for functions of polynomial growth.

Keywords: almost periodic, almost automorphic, ergodic, reduced Beurling spectrum, primitive of weighted almost periodic functions, Esclangon-Landau.

1. Introduction, Notation and Preliminaries

A problem arising naturally from a theorem of Bohl-Bohr-Kadets [21], (see also [4], [9, Sections 5, 6] and references therein) is to investigate the almost periodicity of the primitive $P\psi$ when $\psi = t^N\phi$, where $\phi : \mathbb{R} \to X$ is almost periodic, X is a Banach space, and N is a non-negative integer. More generally we describe the asymptotic behaviour of solutions $u : \mathbb{R} \to X$ of differential equations of the form $\sum_{j=0}^m b_j u^{(j)}(t) = t^N\phi(t)$ where $b_j \in X$ and $m \in \mathbb{N}$.

We begin by introducing some notation. The function $w(t) = w_N(t) = (1 + |t|)^N$ is a weight on $\mathbb{R}$, satisfying in particular $w(s + t) \leq w(s)w(t)$. By J we will mean $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_-$. A function $\phi : J \to X$ is called w-bounded if ϕ/w is bounded and $BC_w(J, X)$ is the space of all continuous w-bounded functions, a Banach space with norm $||\phi||_{w,\infty} = \sup_{t\in\mathbb{R}} \frac{||\phi(t)||}{w(t)}$. Following Reiter [28, p. 142], ϕ is *w-uniformly continuous* if $\|\Delta_h\phi\|_{w,\infty} \to 0$ as $h \to 0$ in J. Here $\Delta_h\phi$ denotes the *difference* of ϕ by h defined by $\Delta_h\phi(t) = \phi(h+t) - \phi(t)$. The closed subspace of $BC_w(J,X)$ consisting of all w-uniformly continuous functions is denoted $BUC_w(J,X)$. It is not hard to show that ϕ is w-uniformly continuous if and only if ϕ/w is uniformly continuous. Furthermore, $\|\phi_{t+h} - \phi_t\|_{w,\infty} \leq w(t)\, \|\phi_h - \phi\|_{w,\infty}$ and so

$$\text{if } \phi \in BUC_w(J,X) \text{ then the function} \\ t \to \phi_t : J \to BUC_w(J,X) \text{ is continuous.} \tag{1.1}$$

When $N = 0$ or equivalently $w = 1$ we will drop the subscript w from the names of various spaces.

As an example, note that for $\lambda, N \neq 0,$ the function $\phi(t) = t^N e^{i\lambda t}$ is not bounded or uniformly continuous. However, ϕ is both w-bounded and w-uniformly continuous and so $\phi \in BUC_w(\mathbb{R}, X)$.

We define $TP_w(\mathbb{R},X) = \operatorname{span}\{t^j e^{i\lambda t}\colon\ 0 \leq j \leq N\ \lambda \in \mathbb{R}\}$ and $AP_w(\mathbb{R},X)$ to be the closure in $BUC_w(\mathbb{R},X)$ of $TP_w(\mathbb{R},X)$.

These are natural generalizations of the spaces $TP(\mathbb{R},X)$ of X-valued trigonometric polynomials and $AP(\mathbb{R},X)$ of almost periodic functions which correspond to the case $N = 0$.

Suppose now that $u' = \psi$ where $u \in BUC_w(\mathbb{R},X)$, $\psi \in AP_w(\mathbb{R},X)$ and $X \not\supseteq c_0$, that is X does not contain a subspace isomorphically isometric to c_0. Kadets proved that necessarily $u \in AP_w(\mathbb{R},X)$ when $N = 0$. However, the following example shows that this is not the case for general N. Indeed, we will show below that the general case is more delicate.

Example 1.1. *Take* $X = \mathbb{C}$, $N = 1$, $w(t) = 1 + |t|$, $\psi(t) = \frac{t}{w(t)} \cos\log w(t)$ *and* $u(t) = \frac{1}{2} w(t)\cos\log w(t) + \frac{1}{2} w(t)\sin\log w(t) - \sin\log w(t) - \frac{1}{2}$. *Then* $u \in BUC_w(\mathbb{R},X)$, $\psi \in AP_w(\mathbb{R},\mathbb{C})$ *and* $u' = \psi$. *However,* $u \notin AP_w(\mathbb{R},\mathbb{C})$.

The proof of this assertion requires some further theory and will be given in Remark 4.4.

2. Some function spaces.

In [12] a function $\phi : J \to X$ is called *Maak-ergodic* with mean $M\phi = x \in X$ (see also [25], [19], [20], [11]) if for each $\varepsilon > 0$ there is a finite subset $F \subseteq J$ with $\|R_F\phi - x)\| < \varepsilon$ where $R_F\phi = \frac{1}{|F|}\sum_{t\in F}\phi_t$. Moreover $E(J,X)$ is the closed subspace of $BC(J,X)$ consisting of Maak-ergodic functions and $E_0(J,X) = \{\phi \in E(J,X) : M\phi = 0\}$. If $M : E(J,X) \to X$ is the function $\phi \to M\phi,$ it follows that M is linear and continuous and $E(J,X) = E_0(J,X) \oplus X$.

In [12] we also defined a notion of ergodicity that applies to unbounded functions. This ergodicity differs from both that of Maak and that of Basit-Günzler [9]. Indeed, the space of w-*ergodic* functions is defined by $E_w(J,X) = \{ \phi \in BC_w(J,X) : \phi/w \in E(J,X)\}$. Both $E_w(J,X)$ and $E_{w,0}(J,X) = \{\phi \in E_w(J,X) : M(\phi/w) = 0\}$ are closed subspaces of $BC_w(J,X)$. It is convenient to introduce an even larger class. For this we need $P_w(J,X)$ the closed subspace of $BC_w(J,X)$ consisting of polynomials on J with coefficients in X. We shall say a function $\phi : J \to X$ is w *-polynomially ergodic* with w*-mean* $p \in P_w(J,X)$ if $(\phi - p)/w \in E_0(J,X)$. The space of all such ϕ is denoted $PE_w(J,X)$ and satisfies $PE_w(J,X) = E_{w,0}(J,X) + P_w(J,X)$. For $N \neq 0$ a w-mean is not unique and this last sum is not direct.

Of course $P_w(J,\mathbb{C})$ is finite dimensional and so $PE_w(J,\mathbb{C})$ is a closed subspace of $BC_w(J,\mathbb{C})$. Moreover, we can choose a subspace $P_w^M(J,\mathbb{C})$ of $P_w(J,\mathbb{C})$ such that $PE_w(J,\mathbb{C}) = E_{w,0}(J,\mathbb{C}) \oplus P_w^M(J,\mathbb{C})$. The (continuous) projection map $M_w : PE_w(J,\mathbb{C}) \to P_w^M(J,\mathbb{C})$ then provides a unique w-polynomial mean $M_w(\phi)$ for each $\phi \in PE_w(J,\mathbb{C})$. Now set $P_w^M(J,X) = P_w^M(J,\mathbb{C}) \otimes X$ and define $M_w : PE_w(J,X) \to P_w^M(J,X)$ by $M_w(\phi) = \sum_{j=1}^k M_w(p_j) \otimes x_j$ where $\phi \in PE_w(J,X)$ has w-polynomial mean $p = \sum_{j=1}^k p_j \otimes x_j \in P_w(J,\mathbb{C}) \otimes X$.

Proposition 2.1. *The map* $M_w : PE_w(J,X) \to P_w^M(J,X)$ *is well-defined and continuous. Moreover, for each* $\phi \in PE_w(J,X)$*,* $M_w(\phi)$ *is a* w*-polynomial mean for* ϕ *and for each of its translates. Finally,* $PE_w(J,X)$ *is a closed translation invariant subspace of* $BC_w(J,X)$ *and* $PE_w(J,X) = E_{w,0}(J,X) \oplus P_w^M(J,X)$.

Proof. Let $\phi \in PE_w(J,X)$ have means $p = \sum_{j=1}^k p_j \otimes x_j$ and $q = \sum_{j=1}^m q_j \otimes y_j$. Then $p - q \in E_{w,0}(J,X)$ and so $x^* \circ (p-q) \in E_{w,0}(J,\mathbb{C})$ for all $x^* \in X^*$. Hence $M_w(x^* \circ (p-q)) = 0 = x^* \circ (\sum_{j=1}^k M_w(p_j) \otimes x_j - \sum_{j=1}^m M_w(q_j) \otimes y_j)$ which gives $\sum_{j=1}^k M_w(p_j) \otimes x_j = \sum_{j=1}^m M_w(q_j) \otimes y_j$ showing M_w is well-defined. Also, $p_j - M_w(p_j) \in E_{w,0}(J,\mathbb{C})$ and so by Lemma 2.2(a) below $p - M_w(p) \in E_{w,0}(J,X)$. Hence $M_w(\phi)$ is a mean for ϕ. Moreover, $\|x^* \circ M_w(\phi)\|_{w,\infty} = \|M_w(x^* \circ \phi)\|_{w,\infty} \leq c\,\|x^* \circ \phi\|_{w,\infty} = c \sup_{t \in J} \|x^* \circ \phi(t)\|/w(t) \leq c\,\|x^*\|\,\|\phi\|_{w,\infty}$. Hence, $|||M_w(\phi)||| \leq c\,\|\phi\|_{w,\infty}$ where $|||\psi||| = \sup_{x^* \in X^*} \frac{\|x^* \circ \psi\|_{w,\infty}}{\|x^*\|}$ for $\psi \in P_w(J,X)$. By Lemma 2.2(b) below, M_w is continuous. If (ϕ_n) is a sequence in $PE_w(J,X)$ converging to ϕ in $BC_w(J,X)$, let $p_n = M_w(\phi_n)$. Then (p_n) converges to some $p \in P_w^M(J,X)$ and so $(\frac{\phi_n - p_n}{w})$ converges to $\frac{\phi - p}{w}$ in $BC(J,X)$. By the continuity of the Maak mean function, $M(\frac{\phi-p}{w}) = 0$ and so $PE_w(J,X)$ is closed. That $PE_w(J,X) =$

$E_{w,0}(J,X) + P_w^M(J,X)$ is clear and that the sum is direct follows from the Hahn-Banach theorem. Finally, for each $t \in J$ we have $\frac{\phi_t - p}{w} = \frac{\Delta_t \phi}{w} + \frac{\phi - p}{w}$ and so, by Lemma 2.2(c) below, p is a w-polynomial mean of ϕ_t and $PE_w(J,X)$ is translation invariant. □

Lemma 2.2.

(a) *If $p \in P_w(J,X)$ and $x^* \circ p \in E_{w,0}(J,\mathbb{C})$ for each $x^* \in X^*$ then $p \in E_{w,0}(J,X)$.*

(b) *On $P_w(J,X)$ the norms $\|\phi\|_{w,\infty}$ and $|||\phi||| = \sup_{x^* \in X^*} \frac{\|x^* \circ \phi\|_{w,\infty}}{\|x^*\|}$ are equivalent.*

(c) *If $\phi \in BC_w(J,X)$ then $\Delta_t \phi \in E_{w,0}(J,X)$ for all $t \in J$.*

(d) *If $\phi \in PE_w(\mathbb{R},X)$ has w-mean p, then $\phi|_J \in PE_w(J,X)$ and $\phi|_J$ has w-mean $p|_J$.*

(e) *$P_w(J,X) \subseteq BUC_w(J,X)$.*

(f) *Let $\phi \in BUC_w(\mathbb{R},X)$, $f \in L_w^1(\mathbb{R})$ and suppose $\phi|_J$ is w-polynomially ergodic with w-mean $p|_J$ where $p \in P_w(\mathbb{R},X)$. Then $(\phi * f)|_J$ is w-polynomially ergodic with w-mean $(p * f)|_J$.*

Proof. (a) We can choose $q_1, ..., q_m \in P_w(J,\mathbb{C})$ and linearly independent unit vectors $x_1, ..., x_m \in X$ such that $p = \sum_{j=1}^m q_j \otimes x_j$. Also choose unit vectors $x_j^* \in X^*$ such that $\langle x_j^*, x_i \rangle = \delta_{i,j}$. Given $\varepsilon > 0$ there are finite subsets F_j of J such that $\left\| R_{Fj}(x_j^* \circ p/w) \right\| < \varepsilon/m$. Setting $F = F_1 + ... + F_m$ we find

$$\begin{aligned}\|R_F(p/w)\| &= \left\| \sum_{j=1}^m R_F(q_j/w) \otimes x_j \right\| \le \sum_{j=1}^m \|R_F(q_j/w)\| \\ &= \sum_{j=1}^m \left\| R_F(x_j^* \circ p/w) \right\| \le \sum_{j=1}^m \left\| R_{F_j}(x_j^* \circ p/w) \right\| < \varepsilon.\end{aligned}$$

This proves that $p \in E_{w,0}(J,X)$

(b) Let $\{p_1, ..., p_k\}$ be a basis of $P_w(J,\mathbb{C})$ consisting of unit vectors for the norm $\|p\|_{w,\infty}$. If $p = \sum_{j=1}^k c_j p_j$, where $c_j \in \mathbb{C}$ then $\|p\|_{w,\infty} \sim \sum_{j=1}^k \|c_j\|$, $\sim$ denoting equivalence of norms. Every $\phi \in P_w(J,X)$ has a unique representation $\phi = \sum_{j=1}^k p_j \otimes x_j$, where $x_j \in X$, and by the closed graph theorem, $\|\phi\|_{w,\infty} \sim \sum_{j=1}^k \|x_j\|$. Hence, $|||\phi||| = \sup \left\| \sum_{j=1}^k p_j \langle x^*, x_j \rangle \right\| / \|x^*\| \le \sum_{j=1}^k \|x_j\| \sim \|\phi\|_{w,\infty}$. Conversely, choose j_0 such that $\|x_j\| \le \|x_{j_0}\|$ for each j. Then choose $x^* \in X^*$ such that $\langle x^*, x_{j_0} \rangle = \|x_{j_0}\|$ and $\|x^*\| = 1$. Hence,

$$
\begin{aligned}
|||\phi||| &\geq \left\| \sum_{j=1}^{k} p_j \left\langle x^*, x_j \right\rangle \right\|_{w,\infty} \\
&\sim \sum_{j=1}^{k} |\langle x^*, x_j\rangle| \geq \|x_{j_0}\| \geq \frac{1}{k} \sum_{j=1}^{k} \|x_j\| \sim \|\phi\|_{w,\infty} .
\end{aligned}
$$

(c) We have $\frac{\Delta_t \phi}{w} = \Delta_t(\frac{\phi}{w}) + (\frac{\phi}{w})_t \frac{\Delta_t w}{w}$ where $\Delta_t(\frac{\phi}{w}) \in E_0(J,X)$ by [11, Proposition 3.2] and $(\frac{\phi}{w})_t \frac{\Delta_t w}{w} \in C_0(J,X)$.

(d) We prove the case $J = \mathbb{R}_+$. Given $\varepsilon > 0$ there is a finite subset $F = \{t_1, ..., t_m\} \subseteq \mathbb{R}$ such that $\left\| \frac{1}{m} \sum_{j=1}^{m} (\frac{\phi - p}{w})(t_j + t) \right\| < \varepsilon$ for all $t \in \mathbb{R}$. choose $u_j, v_j \in \mathbb{R}_+$ such that $t_j = u_j - v_j$. Let $v = v_1 + ... + v_m$ and set $s_j = t_j + v$. So $s_j \in \mathbb{R}_+$ and $\left\| \frac{1}{m} \sum_{j=1}^{m} (\frac{\phi - p}{w})(s_j + t) \right\| < \varepsilon$ for all $t \in \mathbb{R}_+$.

(e) Given $p \in P^n(J,X)$ we may choose $p_j \in P^n(J,X)$ and $q_j \in P^n(J,\mathbb{C})$ with $q_j(0) = 0$ such that $\Delta_h p(t) = \sum_{j=1}^{k} p_j(t) q_j(h)$ for all $h, t \in J$. Hence $\|\Delta_h p(t)\| \leq c w(t) \sum_{j=1}^{k} |q_j(h)|$, where $c = \sup_j \|p_j\|_{w,\infty}$, and so $p \in BUC_w(\mathbb{R},X)$.

(f) If χ is the characteristic function of a compact set $K \subseteq \mathbb{R}$ then $(\phi - p) * \chi(s) = \int_{-K} (\phi - p)_t(s) dt$ for each $s \in \mathbb{R}$. But for each $t \in \mathbb{R}$, $(\phi - p)_t = \Delta_t(\phi - p) + (\phi - p)$ and so by (c), $(\phi - p)_t|_J \in E_{w,0}(J,X)$. Also, by (e), $\phi - p \in BUC_w(\mathbb{R},X)$. By (1.1), the function $t \to (\phi - p)_t|_J : \mathbb{R} \to E_{w,0}(J,X)$ is continuous and hence weakly measurable and separably-valued on $-K$. The integral $\int_{-K} (\phi - p)_t|_J dt$ is therefore a convergent Haar-Bochner integral and so belongs to $E_{w,0}(J,X)$. As evaluation at $s \in J$ is continuous on $E_{w,0}(J,X)$ we conclude that $((\phi - p) * \chi)|_J \in E_{w,0}(J,X)$. Hence also $((\phi - p) * \sigma)|_J \in E_{w,0}(J,X)$ for any step function $\sigma : \mathbb{R} \to \mathbb{C}$ By [28, p. 83] the step functions are dense in $L^1_w(\mathbb{R})$ and so $((\phi - p) * f)|_J \in E_{w,0}(J,X)$ for any $f \in L^1_w(\mathbb{R})$. □

The difference theorem below, included here in order to characterize $E_w(J,X)$ will also be used later. We use the notation $C_{w,0}(J,X) = \{w\xi : \xi \in C_0(J,X)\}$, clearly a closed subspace of $BUC_w(J,X)$.

Theorem 2.3. *Let $\mathcal{F}$ be any translation invariant closed subspace of $BC_w(J,X)$. If $\phi \in PE_w(J,X)$ has w-mean p and $\Delta_t \phi \in \mathcal{F}$ for each $t \in J$, then $\phi - p \in \mathcal{F} + C_{w,0}(J,X)$. If also $w = 1$, then $\phi - p \in \mathcal{F}$.*

Proof. For any finite subset $F \subseteq J$, $\frac{\phi - p}{w} - R_F(\frac{\phi - p}{w}) = -\frac{1}{|F|} \sum_{t \in F} \Delta_t(\frac{\phi - p}{w})$ and so $\phi - p = w R_F(\frac{\phi - p}{w}) - \frac{1}{|F|} \sum_{t \in F} \Delta_t \phi + \frac{1}{|F|} \sum_{t \in F} (\frac{\phi - p}{w})_t \Delta_t w + \frac{1}{|F|} \sum_{t \in F} \Delta_t p$. The first term on the right may be made arbitrarily small in norm by suitable choice of F. The second term is in $\mathcal{F}$ by assumption, the third and fourth terms are in $C_{w,0}(J,X)$ since $\Delta_t p$, $\Delta_t w \in C_{w,0}(J,X)$ for $t \in J$. If $w = 1$ then $\Delta_t w = \Delta_t p = 0$ which shows $\phi - p \in \mathcal{F}$. $\square$

We are now able to characterize w-polynomially ergodic functions. Denote by $D_{w,0}(J,X)$ the closed span of $\{\Delta_t \phi : t \in J, \phi \in BC_w(J,X)\} \cup C_{w,0}(J,X)$ and by $D_w(J,X)$ the closed span of $\{\Delta_t \phi : t \in J, \ \phi \in BC_w(J,X)\}$.

Corollary 2.4. *$E_{w,0}(J,X) = D_{w,0}(J,X)$. If $w = 1$, then $D_{w,0}(J,X) = D_w(J,X) = E_0(J,X)$.*

Proof. Since $C_{w,0}(J,X) \subset E_{w,0}(J,X)$, by Lemma 2.2 (c) and the closedness of $E_{w,0}(J,X)$, we have $D_{w,0}(J,X) \subset E_{w,0}(J,X)$. Conversely, let $\phi \in E_{w,0}(J,X)$. Then $\Delta_t \phi \in D_w(J,X)$ for all $t \in J$ and by Theorem 2.3, $\phi \in D_{w,0}(J,X)$. If $w = 1$, then for any $\psi \in C_{w,0}(J,X)$ and any finite subset F of J, we have $\psi = -\frac{1}{|F|}\sum_{t \in F} \Delta_t \psi + R_F \psi$. As $\|R_F \psi\|_{w,\infty}$ may be made arbitrarily small, we conclude that $\psi \in D_w(J,X)$ and hence $D_{w,0}(J,X) = D_w(J,X)$. $\square$

We conclude this section with a characterization of $AP_w(\mathbb{R},X)$.

Theorem 2.5. *If $N \geq 1$, then $AP_w(\mathbb{R},X) = t^N AP(\mathbb{R},X) \oplus C_{w,0}(\mathbb{R},X)$.*

Proof. Note firstly that the sum on the right is direct. For suppose $t^N \psi_1 + \xi_1 = t^N \psi_2 + \xi_2$ for $\psi_j \in AP(\mathbb{R},X)$ and $\xi_j \in C_{w,0}(\mathbb{R},X)$. Set $q(t) = (1+t)^N - t^N$ and $J = \mathbb{R}_+$. Then $(\psi_1 - \psi_2)|_J = \frac{1}{w}(\xi_2 - \xi_1 - q\psi_2 + q\psi_1)|_J \in C_0(J,X)$. This is impossible unless $\psi_1 = \psi_2$ (see [31] or [5, Proposition 2.1.6). The sum is also topological. For suppose $\phi_n = t^N \psi_n + \xi_n$ where $\psi_n \in AP(\mathbb{R},X)$ and $\xi_n \in C_{w,0}(\mathbb{R},X)$ and (ϕ_n) converges to ϕ in $BC_w(\mathbb{R},X)$. Then $\frac{\phi_n}{w}|_J = \psi_n|_J + \frac{\xi_n - q\psi_n}{w}|_J \in AP(\mathbb{R},X)|_J \oplus C_0(J,X)$. But this last sum is a topological direct sum (see [5, 31]) and so $(\psi_n|_J)$ converges to $\psi|_J$ for some $\psi \in AP(\mathbb{R},X)$. Hence (ψ_n) converges to ψ in $AP(\mathbb{R},X)$ and $\left(t^N \psi_n\right)$ converges to $t^N \psi$ in $AP_w(\mathbb{R},X)$. It follows that (ξ_n) converges to some ξ in $C_{w,0}(\mathbb{R},X)$ and that $\phi = t^N \psi + \xi$.

Next, given $\phi \in AP_w(\mathbb{R},X)$ we may choose a sequence $(\pi_n) \subset TP_w(\mathbb{R},X)$ converging to ϕ in $AP_w(\mathbb{R},X)$. But $\pi_n = t^N \psi_n + \xi_n$ where $\psi_n \in TP(\mathbb{R},X)$ and $\xi_n \in C_{w,0}(\mathbb{R},X)$. It follows from the previous paragraph that $\phi = t^N \psi + \xi$ for some $\psi \in AP(\mathbb{R},X)$ and $\xi \in C_{w,0}(\mathbb{R},X)$.

Conversely, let $\phi = t^N \psi + \xi$ for some $\psi \in AP(\mathbb{R},X)$ and $\xi \in C_{w,0}(\mathbb{R},X)$. We may choose $\alpha_n \in TP(\mathbb{R},X)$ such that $\|\psi(t) - \alpha_n(t)\| \leq \frac{1}{n}$ (see[1, (1.2), p. 15] or [23]). Moreover, $\frac{\xi}{w} \in C_0(\mathbb{R},X)$ and $\|\xi(s)\| \leq$

$\|\xi\|_{\infty,w} w(s)$ for all s. Hence we may choose $\widetilde{t}_n > 0$ such that $\|\xi(t)\| \leq \frac{1}{n}w(t)$ for all $|t| \geq \widetilde{t}_n$. Then choose $t_n > \widetilde{t}_n$ such that $\|\xi\|_{\infty,w} w(\widetilde{t}_n) \leq \frac{1}{n}w(t_n)$. It follows that $\|\xi(t)\| \leq \frac{1}{n}w(t)$ and $\|\xi(s)\| \leq \frac{1}{n}w(t_n)$ for all $|t| \geq t_n$ and all $|s| \leq t_n$. Since ξ is continuous we may choose $\beta_n \in TP(\mathbb{R}, X)$ such that $\|\xi(s) - \beta_n(s)\| \leq \frac{1}{n}$ and $\|\beta_n(t)\| \leq \frac{1}{n}w(t_n) + \frac{1}{n}$ for all $|s| \leq t_n$ and all t. Thus $\|\xi(s) - \beta_n(s)\| \leq \frac{3}{n}w(s)$ for all s. Set $\pi_n = t^N\alpha_n + \beta_n \in TP_w(\mathbb{R}, X)$. Then (π_n) converges to ϕ in $BC_w(\mathbb{R}, X)$ and so $\phi \in AP_w(\mathbb{R}, X)$. □ □

Corollary 2.6. *$AP_w(\mathbb{R}, X) \subset PE_w(\mathbb{R}, X)$.*

Proof. Let $\phi = t^N\psi + \xi$ where $\psi \in AP(\mathbb{R}, X)$ and $\xi \in C_{w,0}(\mathbb{R}, X)$. Set $p = t^N M\psi$ and $\psi_0 = \psi - M\psi$. Then $\frac{\phi - p}{w} = \psi_0 - \frac{\psi_0}{w} + \frac{\xi}{w}$ on $\mathbb{R}_+$ and $\frac{\phi - p}{w} = -\psi_0 + \frac{\psi_0}{w} + \frac{\xi}{w}$ on $\mathbb{R}_-$. Hence $\frac{\phi-p}{w}|_J \in E_0(J, X)$ for $J = \mathbb{R}_+$ or $\mathbb{R}_-$ and thus $\frac{\phi - p}{w} \in E_0(\mathbb{R}, X)$. □

3. Spectral analysis

Throughout this section we will assume that $\mathcal{F}$ is a BUC_w-invariant closed subspace of $BC_w(J, X)$. A subspace $\mathcal{F}$ of $BC_w(J, X)$ is called BUC_w-invariant (see [12]) if $\phi_t|_J \in \mathcal{F}$ whenever $\phi \in BUC_w(\mathbb{R}, X)$, $\phi|_J \in \mathcal{F}$ and $t \in \mathbb{R}$. Numerous examples are provided in [12].

The dual group of $\mathbb{R}$ is denoted $\widehat{\mathbb{R}} = \{\gamma_s : \gamma_s(t) = e^{ist} \text{ for } s, t \in \mathbb{R}\}$ and the Fourier transform of $f \in L^1(\mathbb{R})$ by $\hat{f}(\gamma_s) = \int_{-\infty}^{\infty} f(t)\gamma_s(-t)dt$.

Let $\phi \in BC_w(\mathbb{R}, X)$. The set $I_w(\phi) = \{f \in L^1_w(\mathbb{R}) : \phi * f = 0\}$ is a closed ideal of $L^1_w(\mathbb{R})$ and the *Beurling spectrum* of ϕ is defined to be $sp_w(\phi) = \operatorname{cosp}(I_w(\phi)) = \{\gamma \in \widehat{\mathbb{R}} : \hat{f} = 0 \text{ for all } \gamma \in I_w(\phi)\}$. More generally, following [5, Section 4], the set $I_{\mathcal{F}}(\phi) = \{f \in L^1_w(\mathbb{R}) : (\phi * f)|_J \in \mathcal{F}\}$ is a closed translation invariant subspace of $L^1_w(\mathbb{R})$ and therefore an ideal. We define the *spectrum of ϕ relative to $\mathcal{F}$*, or the *reduced Beurling spectrum*, to be $sp_{\mathcal{F}}(\phi) = \operatorname{cosp}\,(I_{\mathcal{F}}(\phi))$.

The following proposition contains some basic properties of these spectra. The proofs are the same as for the Beurling spectrum. See for example [17, p. 988] or [29] also [6], [15], [27].

Proposition 3.1. *Let $\phi, \psi \in BC_w(\mathbb{R}, X)$.*

(a) *$sp_{\mathcal{F}}(\phi_t) = sp_{\mathcal{F}}(\phi)$ for all $t \in \mathbb{R}$.*

(b) *$sp_{\mathcal{F}}(\phi * f) \subseteq sp_{\mathcal{F}}(\phi) \cap supp(\hat{f})$ for all $f \in L^1_w(\mathbb{R})$.*

(c) *$sp_{\mathcal{F}}(\phi + \psi) \subseteq sp_{\mathcal{F}}(\phi) \cup sp_{\mathcal{F}}(\psi)$.*

(d) *$sp_{\mathcal{F}}(\gamma\phi) = \gamma sp_{\mathcal{F}}(\phi)$, provided $\mathcal{F}$ is invariant under multiplication by $\gamma \in \widehat{\mathbb{R}}$.*

(e) *If* $f \in L^1_w(\mathbb{R})$ *and* $\hat{f} = 1$ *on a neighbourhood of* $sp_{\mathcal{F}}(\phi)$*, then* $sp_{\mathcal{F}}(\phi * f - \phi) = \emptyset$.

The following theorem is proved in [12](see also [10], [11]). It gives our motivation for introducing $sp_{\mathcal{F}}(\phi)$.

Theorem 3.2. *Let* $\phi \in BUC_w(\mathbb{R}, X)$.

(a) *If* $f \in L^1_w(G)$ *and* $\phi|_J \in \mathcal{F}$*, then* $(\phi * f)|_J \in \mathcal{F}$.

(b) $sp_{\mathcal{F}}(\phi) = \emptyset$ *if and only if* $\phi|_J \in \mathcal{F}$.

(c) *If* $\Delta^k_t \phi|_J \in \mathcal{F}$ *for all* $t \in \mathbb{R}$ *and some* $k \in \mathbb{N}$*, then* $sp_{\mathcal{F}}(\phi) \subseteq \{1\}$.

(d) $sp_{\mathcal{F}}(\phi) \subseteq \{\gamma_1, ..., \gamma_n\}$ *if and only if* $\phi = \psi + \sum_{j=1}^n \eta_j \gamma_j$ *for some* $\psi, \eta_j \in BUC_w(\mathbb{R}, X)$ *with* $\psi|_J \in \mathcal{F}$ *and* $\Delta_t \eta_j|_J \in \mathcal{F}$ *for each* $t \in \mathbb{R}^{N+1}$.

4. Primitives and Derivatives

Throughout this section we assume that $\mathcal{F}$ is a translation invariant closed subspace of $BUC_w(J, X)$. Examples of such classes are $P_w(J, X)$, $C_{w,0}(J, X)$, $AP_w(\mathbb{R}, X)$, $E_{w,0}(J, X) \cap BUC_w(J, X)$ and $PE_w(J, X) \cap BUC_w(J, X)$.

We define the *primitive* $P\phi$ of a function $\phi \in BC_w(\mathbb{R}, X)$ by $P\phi(t) = \int_0^t \phi(s)ds$.

Theorem 4.1.

(a) *If* $\mathcal{F}_w$ *denotes any of* $BC_w(J, X)$, $C_{w,0}(J, X)$, $E_{w,0}(J, X)$, $P_w(J, X)$, $PE_w(J, X)$ *or* $AP_w(\mathbb{R}, X)$ *then* P *maps* $\mathcal{F}_w$ *continuously into* $\mathcal{F}_{ww_1}$.

(b) *If* $\phi \in E_{w,0}(J, X)$ *then* $P\phi \in C_{ww_1,0}(J, X)$.

(c) *If* $\phi \in AP_w(J, X)$ *has* w*-mean* p*. Then* $P(\phi - p) \in C_{ww_1,0}(\mathbb{R}, X)$.

Proof. Take $J = \mathbb{R}_+$, the other cases being proved similarly. If $\phi \in BC_w(J, X)$ and $t \in J$ then $||P\phi(t)|| \le t.||\phi||_{w,\infty} w(t)$. Hence P maps $BC_w(J, X)$ continuously into $BC_{ww_1}(J, X)$. If also $\phi \in C_{w,0}(J, X)$ then given $\varepsilon > 0$ there exists $t_0 > 0$ such that $||\phi(t)|| < \varepsilon w(t)$ whenever $t > t_0$. For these t we have $||P\phi(t)|| \le \int_0^{t_0} ||\phi(s)||\, ds + \varepsilon w(t)(t - t_0)$ and so P maps $C_{w,0}(J, X)$ into $C_{ww_1,0}(J, X)$. Next, $P(\Delta_t \phi) = \Delta_t(P\phi) - P\phi(t)$ and since P is continuous it follows from Corollary 2.4 that P maps $E_{w,0}(J, X)$ into $E_{ww_1,0}(J, X)$. The result for $P_w(J, X)$ is clear and so therefore is the result for $PE_w(J, X)$. For (b) note that $||\Delta_t\, P\phi(s)|| \le t.w(t)w(s)\, ||\phi||_{w,\infty}$ for all $s \in J$. Hence $\Delta_t(P\phi) \in C_{ww_1,0}(J, X)$. If $\phi \in E_{w,0}(J, X)$, we can apply Theorem 2.3 to $P\phi$ to obtain $P\phi \in C_{ww_1,0}(J, X)$. Finally, (c) follows from (b) using Corollary 2.6, and then (a) with $\mathcal{F}_w = AP_w(\mathbb{R}, X)$ follows from (c) using Theorem 2.5. $\square$

Proposition 4.2.

(a) *If* $\phi \in BC_w(\mathbb{R}, X)$ *and* $sp_w(\phi)$ *is compact, then* $\phi^{(j)} \in BUC_w(\mathbb{R}, X)$ *for all* $j \geq 0$.

(b) *If* $\phi \in \mathcal{F}$ *and* ϕ' *is* w*-uniformly continuous, then* $\phi' \in \mathcal{F}$.

(c) *If* $\phi \in BC_w(J, X)$ *and* ϕ' *is* w *-uniformly continuous, then* $\phi' \in E_{w,0}(J, X) \cap BUC_w(J, X)$.

(d) *If* $\phi, \phi' \in BC_w(\mathbb{R}, X)$ *then* $sp_w(\phi') \subseteq sp_w(\phi) \subseteq sp_w(\phi') \cup \{1\}$.

(e) *If* $\phi, \phi' \in BC_w(J, X)$ *then* $\phi \in BUC_w(J, X)$.

Proof. (a) Choose $f \in S(\mathbb{R})$, the Schwartz space of rapidly decreasing functions, such that f has compact support and is 1 on a neighbourhood of $sp_w(\phi)$. Then $f^{(j)} \in L^1_w(\mathbb{R})$ for all $j \geq 0$. Moreover, $\phi = \phi * f$ and so $\phi^{(j)} = \phi * f^{(j)}$ for all $j \geq 0$. Hence $\phi^{(j)} \in BUC_w(\mathbb{R}, X)$.

(b) If $\psi_n = n\Delta_{1/n}\phi$ then $\psi_n \in \mathcal{F}$. Moreover, by the w-uniform continuity of ϕ', given $\varepsilon > 0$ there exists n_ε such that

$$\|\psi_n(t) - \phi'(t)\| = \left\| n \int_0^{1/n} (\phi'(t+s) - \phi'(t))ds \right\| < \varepsilon w(t)$$

for all $t \in J$ and $n > n_\varepsilon$. Hence $\phi' \in \mathcal{F}$.

(c) With the notation used in the proof of (b), $\psi_n \in E_{w,0}(J, X) \cap BUC_w(J, X)$ by Lemma 2.2(c). Hence, so does ϕ'.

(d) For any $f \in L^1_w(\mathbb{R})$ we have $(\phi * f)' = \phi' * f$ and so $I_w(\phi') \supseteq I_w(\phi)$. Hence, $sp_w(\phi') \subseteq sp_w(\phi)$. For the second inclusion, let $g(t) = \exp(-t^2)$ so that $g, g' \in L^1_w(\mathbb{R})$ and $\hat{g}$ is never zero. Now take $\gamma \in \widehat{\mathbb{R}}\ (sp_w(\phi') \cup \{1\})$. So $\gamma(t) = e^{ist}$ for some $s \neq 0$ and there exists $f \in L^1_w(\mathbb{R})$ such that $\phi' * f = 0$ but $\hat{f}(\gamma) \neq 0$. Let $h = f * g' \in L^1_w(\mathbb{R})$. Then $\phi * h = \phi * f * g' = \phi' * f * g = 0$ whereas $\hat{h}(\gamma) = is\hat{f}(\gamma)\hat{g}(\gamma) \neq 0$. So $\gamma \notin sp_w(\phi')$ showing $sp_w(\phi) \subseteq sp_w(\phi') \cup \{1\}$.

(e) For any $h, t \in J$ we have $\|\Delta_h\phi(t)\| = \left\| \int_t^{t+h} \phi'(s)ds \right\| \leq |h| \cdot \|\phi'\|_{w,\infty} w(h)w(t)$ from which it follows that ϕ is w-uniformly continuous. □

Proposition 4.3. *Let* $\phi \in \mathcal{F}$ *and assume that* $\mathcal{F}$ *is* BUC_w*-invariant.*

(a) *If* $P\phi$ *is* w*-polynomially ergodic with* w*-mean* p*, then* $P\phi - p \in \mathcal{F} + C_{w,0}(J, X)$.

(b) *If* $\mathcal{F} = AP_w(\mathbb{R}, X)$ *and* $P\phi \in PE_w(\mathbb{R}, X)$*, then* $P\phi \in AP_w(\mathbb{R}, X)$.

(c) *If* $\mathcal{F} = AP_w(\mathbb{R}, X)$ *and* $P\phi \in BUC_w(\mathbb{R}, X)$*, then* $sp_{\mathcal{F}}(P\phi) \subseteq \{1\}$.

(d) *If* $\mathcal{F} = C_0(\mathbb{R}_+, X)$ *and* $P\phi$ *is ergodic with mean* c*, then* $P\phi - c \in C_0(\mathbb{R}_+, X)$.

Proof. (a) Take $J=\mathbb{R}_+,$ the other cases being proved similarly. Extend ϕ to an even function $\widetilde{\phi}\in BUC_w(\mathbb{R},X)$. For $t\geq 0$ set $\chi_t=\chi_{[-t,0]}$ so that $\Delta_t P\phi=\left(\widetilde{\phi}*\chi_t\right)|_J=\int_{\mathbb{R}}\left(\widetilde{\phi}_{-s}\right)|_J\chi_t(s)ds$. Since $\widetilde{\phi}\in BUC_w(\mathbb{R},X)$ the integral converges as a Lebesgue-Bochner integral. Since $\mathcal{F}$ is BUC_w-invariant $\left(\widetilde{\phi}_{-s}\right)|_J\in\mathcal{F}$ and therefore $\Delta_t P\phi\in\mathcal{F}$. The result follows from Theorem 2.3.

(b) In view of Theorem 2.5, this follows from (a).

(c) Let $s,t\in\mathbb{R}$ With χ_s as in the previous proof, $(\Delta_s P\phi)_t=\phi_t*\chi_s$ and by Proposition 3.2 (a), $(\Delta_s P\phi)_t\in\mathcal{F}$. By Proposition 3.2(c), $sp_{\mathcal{F}}(P\phi)\subseteq\{1\}$.

(d) This is a special case of part (a). □

Remark 4.4. *Recall* $u(t)=\frac{1}{2}w(t)\cos\log w(t)+\frac{1}{2}w(t)\sin\log w(t)-\sin\log w(t)-\frac{1}{2}$ *from Example 1.1. So* $u'(t)=\frac{t}{w(t)}\cos\log w(t)$ *and therefore* $u'\in C_{w,0}(\mathbb{R},\mathbb{C})\subset AP_w(\mathbb{R},\mathbb{C})\subset PE_w(\mathbb{R},\mathbb{C})$. *However,* $u\notin PE_w(\mathbb{R},\mathbb{C})$. *Indeed, if* $u\in PE_w(\mathbb{R},\mathbb{C})$ *then for* $t\in\mathbb{R}_+$ *set* $\xi(t)=w(t)\cos\log w(t)+w(t)\sin\log w(t)$. *So* $\xi\in PE_w(\mathbb{R}_+,\mathbb{C})$ *and for some polynomial* $p(t)=at+b$ *we have* $(\xi-p)/w\in E_0(\mathbb{R}_+,\mathbb{C})$. *Thus* $\eta=\xi/w\in E(\mathbb{R}_+,\mathbb{C})$. *But* $\eta'(t)=[-\sin\log w(t)+\cos\log w(t)]/w(t)$ *and so* $\eta'\in C_0(\mathbb{R},\mathbb{C})$. *By Proposition 4.3(d) we conclude* $\eta\in C_0(\mathbb{R}_+,\mathbb{C})+\mathbb{C}$ *which is false.*

Lemma 4.5. *For natural numbers* m,N *and non-negative integers* j,k *set* $a(m,j)=(-1)^j\binom{N}{j}\binom{m-1+j}{j}j!$.

(a) $P^m(t^N\phi)=\sum_{j=0}^N a(m,j)\,t^{N-j}P^{m+j}\phi$ *for any* $\phi\in L^1_{loc}(J,X)$.

(b) $$\sum_{j=0}^N\frac{a(m,j)}{(j+k)!}=\begin{cases}\binom{N+k-m}{N}\frac{N!}{(N+k)!} & \text{if } m\leq k\\ 0 & \text{if } k+1\leq m\leq k+N\,.\\ (-1)^N\binom{m-k-1}{N}\frac{N!}{(N+k)!} & \text{if } m>k+N\end{cases}$$

(c) $\sum_{j=0}^N a(m,j)\,t^{N-j}P^{j+1}r=P\sum_{j=0}^N a(m-1,j)\,t^{N-j}P^j\,r$ *for any* $r\in P_{m-2}(J,X)$.

Proof. (a) For $N=1$ the claim is readily proved by induction on m. The general case is then proved by induction on N.

(b) For $m\geq k+1$ we have

$$\sum_{j=0}^N\frac{a(m,j)}{(j+k)!}=\sum_{j=0}^N(-1)^j\binom{N}{j}\binom{m-1+j}{j}\frac{j!}{(j+k)!}$$

$$= \frac{1}{(m-1)!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(m-1+j)!}{(k+j)!}$$
$$= \frac{1}{(m-1)!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} D^{m-k-1} t^{m+j-1}|_{t=1}$$
$$= \frac{1}{(m-1)!} D^{m-k-1} t^{m-1} \sum_{j=0}^{N} (-1)^j \binom{N}{j} t^j|_{t=1}$$
$$= \frac{1}{(m-1)!} D^{m-k-1}\ t^{m-1}(1-t)^N|_{t=1}.$$

For $m-k-1 < N$ this last expression is 0 and for $m-k-1 \geq N$ it is

$$\frac{1}{(m-1)!} \binom{m-k-1}{N} (D^{m-k-1-N}\ t^{m-1}) D^N (1-t)^N|_{t=1}$$
$$= \frac{1}{(m-1)!} \binom{m-k-1}{N} \frac{(m-1)!}{(N+k)!} (-1)^N N!$$

as claimed. For $m \leq k$ the claim follows readily by substituting $\phi(t) = t^{k-m}$ in (a).

(c) It follows readily from (b) that $\sum_{j=0}^{N} \frac{a(m-1,j)}{(j+k)!} = (N+k+1) \times \sum_{j=0}^{N} \frac{a(m,j)}{(j+k+1)!}$ if $0 \leq k \leq m-2$. So setting $r(t) = \sum_{k=0}^{m-2} c_k t^k$ we find

$$\sum_{j=0}^{N} a(m,j)\ t^{N-j}\ P^{j+1} r(t)$$
$$= \sum_{k=0}^{m-2} c_k k! t^{N+k+1} \sum_{j=0}^{N} \frac{a(m,j)}{(j+k+1)!}$$
$$= \sum_{k=0}^{m-2} c_k k! t^{N+k+1} \frac{1}{N+k+1} \sum_{j=0}^{N} \frac{a(m-1,j)}{(j+k)!}$$
$$= P \sum_{k=0}^{m-2} c_k k! t^{N+k} \sum_{j=0}^{N} \frac{a(m-1,j)}{(j+k)!}$$
$$= P \sum_{j=0}^{N} a(m-1,j)\ t^{N-j} P^j r(t). \qquad \square$$

Our main result is the following:

Theorem 4.6. *Assume $\phi \in AP(\mathbb{R}, X)$ and that $\sum_{j=0}^{N} b_j\, t^{N-j} P^{j+1} \phi \in BUC_{w_N}(\mathbb{R}, X)$ for some $b_j \in \mathbb{C}$, $b_0 \neq 0$.*

(a) $P\phi \in BUC(\mathbb{R}, X)$ *and if* $\sum_{j=0}^{N} \frac{b_j}{(j+1)!} \neq 0$ *then* $M\phi = 0$.

(b) *If* $X \not\supseteq c_0$ *then* $P(\phi - M\phi) \in AP(\mathbb{R}, X)$.

Proof. Let $a = M\phi$ and $\psi = \sum_{j=0}^{N} b_j\, t^{N-j} P^{j+1}\phi$. Then we have $\psi = \sum_{j=0}^{N} b_j t^{N-j} P^{j+1}(\phi - a) + t^{N+1} a \sum_{j=0}^{N} \frac{b_j}{(j+1)!}$. By Theorem 4.1(c), $\psi - \sum_{j=0}^{N} b_j\, t^{N-j} P^{j+1}(\phi - a) \in C_{w_{N+1},0}(\mathbb{R}, X)$ and so either $a = 0$ or $\sum_{j=0}^{N} \frac{b_j}{(j+1)!} = 0$. To prove the rest of the theorem, we may assume $a = 0$. By Theorem 4.1(c), $P^j\phi(t)/w_j(t) \to 0$ as $t \to \infty$. Since ϕ is almost periodic we may choose $(t_n) \subset \mathbb{R}$ such that $t_n \to \infty$ and $\phi_{t_n} \to \phi$ uniformly on $\mathbb{R}$. Moreover, as $M\phi = 0$, by Theorem 4.1(c), $P^j\phi(s+t_n)/w_j(s+t_n) \to 0$ uniformly on $\mathbb{R}$ for $j > 0$. Given $x^* \in X^*$, it follows that $x^* \circ P\phi_{t_n} \to x^* \circ P\phi$ locally uniformly. Moreover, by passing to a subsequence if necessary, we may assume $x^* \circ \psi(t_n)/w_N(t_n) \to b$ for some $b \in \mathbb{C}$. By Theorem 4.1(c) again, we obtain

$$\begin{aligned}\psi(t + t_n) &= \sum_{j=0}^{N} b_j\, (t + t_n)^{N-j} [\int_0^t P^j\phi(s + t_n) ds + P^{j+1}\phi(t_n)] \\ &= \psi(t_n) + b_0 t_n^N P\phi(t + t_n) + o(t_n^N).\end{aligned}$$

Therefore $x^* \circ \psi(t + t_n)/w_N(t + t_n) \to b + b_0 x^* \circ P\phi(t)$ for each $t \in \mathbb{R}$. Hence, since ψ/w_N is bounded, so too is $x^* \circ P\phi$. Since x^* is arbitrary, $P\phi$ is weakly bounded and therefore bounded. From Proposition 4.2(e) it follows that $P\phi \in BUC(\mathbb{R}, X)$. If also $X \not\supseteq c_0$ then by Kadet's theorem [21] (see also [4]), $P\phi$ is almost periodic. □

Corollary 4.7. *Assume* $\phi \in AP(\mathbb{R}, X)$ *and* $P(t^N\phi) \in BUC_{w_N}(\mathbb{R}, X)$.

(a) $P\phi \in BUC(\mathbb{R}, X)$ *and* $M\phi = 0$.

(b) *If* $X \not\supseteq c_0$ *then* $P\phi \in AP(\mathbb{R}, X)$.

Proof. Since $P^m(t^N\phi) = \sum_{j=0}^{N} a(m, j)\, t^{N-j} P^{m+j}\phi$ the result follows from Theorem 4.6 and Lemma 4.5. □

Theorem 4.8. *Assume* $\phi \in AP(\mathbb{R}, X)$, $X \not\supseteq c_0$, $P^m(t^N\phi) + p \in BUC_{w_N}(\mathbb{R}, X)$ *for natural numbers* m, N *and some* $p \in P_{m-1}(\mathbb{R}, X)$. *Then* $P^j(t^N\phi) + p^{(m-j)} \in AP_{w_N}(\mathbb{R}, X)$ *for* $1 \le j \le m$. *Moreover, there is a polynomial* $q \in P_{m-1}(\mathbb{R}, X)$ *such that* $P^j\phi + q^{(m-j)} \in AP(\mathbb{R}, X)$ *for* $1 \le j \le m$ *and, if* $p(t) = \sum_{k=0}^{m-1} b_k t^k$ *then* $\sum_{j=0}^{N} a(m, j)\, t^{N-j} P^j q = \sum_{k=N+1}^{m-1} b_k t^k$.

Proof. The proof is by induction on m. If $m = 1$ then, by Lemma 4.5 $\sum_{j=0}^{N} \frac{a(1,j)}{(j+1)!} = \frac{1}{N+1}$ and $P(t^N\phi) = \sum_{j=0}^{N} a(1, j)\, t^{N-j} P^{j+1}\phi \in$

$BUC_{w_N}(\mathbb{R},X)$. Therefore, by Theorem 4.6, $M\phi = 0$ and $P\phi \in AP(\mathbb{R},X)$. Moreover, by Lemma 4.5(b),

$$\sum_{j=0}^{N} a(1,j)\, t^{N-j} P^j (MP\phi) \;=\; (MP\phi)\, t^N \;\sum_{j=0}^{N} \frac{a_j}{j!} = 0.$$

Hence $P(t^N\phi) = \sum_{j=0}^{N} a(1,j)\, t^{N-j} P^j(P\phi - MP\phi)$ and by Theorem 4.1(c), $P(t^N\phi) \in AP_{w_N}(\mathbb{R},X)$. For $m > 1$, Theorem 5.2 below shows $P^j(t^N\phi) + p^{(m-j)} \in BUC_{w_N}(\mathbb{R},X)$ for $1 \le j \le m$. Hence, as induction hypothesis we may assume there is a polynomial $r \in P_{m-2}(\mathbb{R},X)$ such that for $1 \le j \le m-1$ we have $P^j\phi + r^{(m-1-j)} \in AP(\mathbb{R},X)$, $P^j(t^N\phi) + p^{(m-j)} \in AP_{w_N}(\mathbb{R},X)$ and $\sum_{j=0}^{N} a(m-1,j)\, t^{N-j} P^j r = \sum_{k=N+2}^{m-1} k b_k t^{k-1}$. In particular, $\eta = P^{m-1}\phi + r + c \in AP(\mathbb{R},X)$ where the constant c is to be chosen. Moreover, by Lemma 4.5(d), $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1} r = \sum_{k=N+2}^{m-1} b_k t^k$.

Now set $q = P(r + c - M\phi)$ so that $P^m\phi + q = P(\eta - M\phi)$. By Theorem 4.6, to show $P^m\phi + q \in AP(\mathbb{R},X)$, it suffices to show that $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1}\eta \in BUC_{w_N}(\mathbb{R},X)$. By Lemma 4.5(a),

$$\begin{aligned} P^m(t^N\phi) &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{m+j}\phi \\ &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1}(\eta - r - c). \end{aligned}$$

Since $P^m(t^N\phi) + p \in BUC_{w_N}(\mathbb{R},X)$, it suffices to show $\sum_{j=0}^{N} \{a(m,j) \times t^{N-j} P^{j+1}(r + c)\} = \sum_{k=N+1}^{m-1} b_k t^k$.

If $N > m - 2$ we choose $c = 0$ as then both sides are 0. Otherwise $N \le m - 2$ and by Lemma 4.5(c) we may choose c such that $\sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1} c = b_{N+1} t^{N+1}$, that is $c \sum_{j=0}^{N} \frac{a(m,j)}{(j+1)!} = b_{N+1}$. In this case also we have by Theorem 4.6, $M\eta = 0$. In either case $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^j q = \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1}(r + c) = \sum_{k=N+1}^{m-1} b_k t^k$ and $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^{j+1} M\eta = 0$. Finally,

$$\begin{aligned} P^m(t^N\phi) + p &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^{j+1}(\eta - r - c - M\phi) + p \\ &= \sum_{j=0}^{N} a(m,j)\; t^{N-j} P^j(P^m\phi + q) + \sum_{k=0}^{N} b_k t^k \end{aligned}$$

and by Theorem 4.1, $P^m(t^N\phi) + p \in AP_{w_N}(\mathbb{R},X)$. $\square$

Remark 4.9. (a) *In Theorem 4.6(a) the space $AP(\mathbb{R},X)$ may be replaced by the class of Poisson stable functions. These are functions $\xi \in C(\mathbb{R},X)$ for which there exist sequences $(t_n) \subset \mathbb{R}$ such that $t_n \to \infty$ and $\xi_{t_n} \to \xi$ locally uniformly on $\mathbb{R}$. In part (b), $AP(\mathbb{R},X)$ may be replaced by any class for which Kadet's theorem remains valid. These include Poisson stable functions, almost automorphic functions and recurrent functions (see [4]).*

(b) *If $p = 0$ in Theorem 4.8 then $\sum_{j=0}^{N} a(m,j)\, t^{N-j} P^j q = 0$, which reduces to $q^{(k)}(0) = 0$ for $0 \le k \le m - N - 1$.*

(c) *Assume $\phi \in AP(\mathbb{R},X)$ where $X \not\supseteq c_0$. By Theorem 4.8, if $P^m(t^N\phi) + p \in BUC_{w_N}(\mathbb{R},X)$ for some p then $P^m\phi + q \in AP(\mathbb{R},X)$ for some q.*

The converse is also true. Indeed, $P^m(t^N\phi) = \sum_{j=1}^{N} \{a(m,j)\times t^{N-j}P^{m+j}\phi\} + t^N P^m\phi$ and the result follows from Theorem 4.1(c).

(d) *These results are dependent on the Poisson stability property of ϕ. Indeed, consider the function $\phi \in C_0(\mathbb{R},\mathbb{C})$ given by $\phi(t) = \frac{1}{1+|t|}$. Then $P(t\phi) = |t| - \ln(1+|t|)$ and $P\phi = sgn(t)\ln(1+|t|)$. Hence $P(t\phi) \in BUC_{w_1}(\mathbb{R},\mathbb{C})$ whereas $P\phi \notin BC(\mathbb{R},\mathbb{C})$.*

(e) *A well-known example to show that the condition $X \not\supseteq c_0$ may not be omitted from Theorem 4.6 is as follows. Let $X = c_0$ and $\phi(t) = (\frac{1}{n}\sin\frac{t}{n})_{n=1}^{\infty}$ so that $P\phi(t) = (2\sin^2\frac{t}{2n})_{n=1}^{\infty}$. Then $\phi \in AP(\mathbb{R},c_0)$ and $P\phi \in BUC(\mathbb{R},c_0)$. However, $P\phi$ does not have relatively compact range so it is not almost periodic.*

5. Esclangon-Landau theorem

In this section we use the abbreviations

$$Bu = \sum_{j=0}^{m} b_j u^{(j)} \tag{5.2}$$

and assume $b_m = 1$, $b_j \in \mathbb{C}$, $u : J \to X$.

We prove a theorem of Esclangon-Landau type ([18], [22], [14], [7], [16] and references therein).

Lemma 5.1. *If $Bu = \psi$ where $u, \psi \in BC_{w_N}(J,X)$ then $u^{(j)}(t) = O(|t|^{N+m-1})$ for $1 \le j \le m$.*

Proof. Since $u^{(m)} = \psi - \sum_{j=0}^{m-1} b_j u^{(j)}$, taking P^{m-k} we obtain

$$u^{(k)} = \sum_{j=1}^{m-k} P^{j-1}(u^{(j+k-1)}(0)) + P^{m-k}\psi - \sum_{j=0}^{m-1} b_j P^{m-k} u^{(j)}.$$

Setting $k = 1$ we conclude that $u'(t) = O(|t|^{N+m-1})$. In general $u^{(k)}(t) = O(|t|^{N+m-1}) - \sum_{j=m-k+1}^{m-1} b_j P^{m-k} u^{(j)}(t) = O(|t|^{N+m-1}) + \sum_{j=1}^{k-1} O(|u^{(j)}(t)|)$ from which the result follows by induction. $\square$

Theorem 5.2. *If $Bu = \psi$ where $u, \psi \in BC_{w_N}(J, X)$ then $u^{(j)} \in BC_{w_N}(J, X)$ for $1 \leq j \leq m$.*

Proof. Take $J = \mathbb{R}_+$, the other cases being proved similarly. The proof is by induction on m. First, if $m = 1$ the equation becomes $u' + b_0 u = \psi$ showing $u' \in BC_{w_N}(J, X)$. For the general case we use the functions f and $\tilde{u}$ defined by $f(t) = \exp(-t)$ for $t \geq 0$, $f(t) = 0$ for $t < 0$, $\tilde{u}(t) = u(-t)$ for $-t \in J$ and $\tilde{u}(t) = 0$ for $-t \notin J$. It follows that $e^t \int_t^\infty e^{-s} u(s) ds = \int_0^\infty e^{-s} u(s + t) ds = \tilde{u} * f(-t)$ and $\tilde{u} * f \in BC_{w_N}(\mathbb{R}, X)$. Moreover, using repeated integration by parts and Lemma 5.1, we find $e^t \int_t^\infty e^{-s} u^{(k)}(s) ds = -\sum_{j=1}^{k-1} u^{(j)}(t) + \tilde{u} * f$ $(-t)$. Hence the equation $B\phi = \psi$ may be transformed to the equation $\sum_{k=1}^m b_k \sum_{j=1}^{k-1} u^{(j)}(t) = (\sum_{k=0}^m b_k) \tilde{u} * f\ (-t) - \tilde{\psi} * f\ (-t)$. This is an equation of order $m - 1$ and so by the induction hypothesis $u^{(j)} \in BC_{w_N}(J, X)$ for $1 \leq j \leq m - 1$. Hence $u^{(m)} = \psi - \sum_{j=0}^{m-1} b_j u^{(j)} \in BC_{w_N}(J, X)$ which finishes the proof. $\square$

6. Application

Again we use the abbreviation $Bu = \sum_{j=0}^m b_j u^{(j)}$ and assume $b_m = 1$. By p_B we denote the characteristic polynomial of the differential operator B. Thus $p_B(s) = \sum_{j=0}^m b_j (is)^j$ and for smooth f we have $\widehat{Bf}(\gamma_s) = p_B(s) \hat{f}(\gamma_s)$. The set of complex zeros of p_B is denoted $Z(B)$.

Lemma 6.1. *Assume $u \in BC_{w_N}(\mathbb{R}, X)$ and $\mathcal{F}$ is a BUC_{w_N}-invariant closed subspace of $BC_{w_N}(J, X)$. If $Bu = \psi$ where $\psi \in BUC_{w_N}(\mathbb{R}, X)$ and $\psi|_J \in \mathcal{F}$ then $sp_{\mathcal{F}}(u) \subset \{\gamma_s : s \in Z(B) \cap \mathbb{R}\}$.*

Proof. Take $s \in \mathbb{R}$ with $p_B(s) \neq 0$. Choose $f \in S(\mathbb{R})$ with $\hat{f}(\gamma_s) \neq 0$ and set $g = Bf$. Then $u * g = \psi * f$ and by Theorem 3.2(a), $(\psi * f)\,|_J \in \mathcal{F}$. Hence $g \in I_{\mathcal{F}}(u)$ whereas $\hat{g}(\gamma_s) = p_B(s) \hat{f}(\gamma_s) \neq 0$. So $\gamma_s \notin sp_{\mathcal{F}}(u)$ and the proof is completed. $\square$

Theorem 6.2. *Suppose $Bu = \psi$ where $u \in BC_{w_N}(\mathbb{R}, X)$ and $\psi \in AP_{w_N}(\mathbb{R}, X)$.*

(a) *If $Z(B) \cap \mathbb{R} = \varnothing$ then $u^{(j)} \in AP_{w_N}(\mathbb{R}, X)$ for $0 \leq j \leq m$.*

(b) *If $Z(B) \cap \mathbb{R} \neq \varnothing$, but $X \not\supseteq c_0$ and $\psi = t^N \phi$ where $\phi \in AP(\mathbb{R}, X)$ then $u^{(j)} \in AP_{w_N}(\mathbb{R}, X)$ for $0 \leq j \leq m$.*

Proof. (a) Let $\mathcal{F} = AP_{w_N}(\mathbb{R}, X)$. By Lemma 6.1, $sp_{\mathcal{F}}(u) \subset Z(B) \cap \mathbb{R} = \varnothing$. Hence, by Theorem 3.2(b), $u \in \mathcal{F}$. The Esclangon-Landau Theorem 5.2 shows $u, u', ..., u^{(m)} \in BC_{w_N}(\mathbb{R}, X)$ and then Proposition 4.2(e) shows $u, u', ..., u^{(m-1)} \in BUC_{w_N}(\mathbb{R}, X)$. From Proposition 4.2(b) we conclude $u', ..., u^{(m-1)} \in \mathcal{F}$. Rearranging the differential equation, we obtain $u^{(m)} \in \mathcal{F}$.

(b) The proof is by induction on m. Note first that by Theorem 5.2, $u^{(j)} \in BC_{w_N}(\mathbb{R}, X)$ for $0 \le j \le m$. Let $\lambda \in Z(B) \cap \mathbb{R}$ and make the substitution $\eta(t) = \exp(-i\lambda t)u(t)$ so that $\eta^{(j)} \in BC_{w_N}(\mathbb{R}, X)$ for $0 \le j \le m$. If $m = 1$ the equation $u' - i\lambda u = t^N \phi$ reduces to $\eta' = \exp(-i\lambda t)t^N \phi$. From Theorem 4.8 we conclude $\eta \in \mathcal{F}$. Hence $u, u' \in \mathcal{F}$ as claimed. For general m, the equation $Bu = t^N \phi$ reduces to an equation of the form $\sum_{j=1}^{m} c_j \eta^{(j)} = \exp(-i\lambda t)t^N \phi$ where $c_m = 1$. This is a differential equation in η' of order $m - 1$. By the induction hypothesis, or by part (a) if the characteristic polynomial has no real zeros, $\eta^{(j)} \in \mathcal{F}$ for $1 \le j \le m - 1$. It remains to show $\eta \in \mathcal{F}$. For this, let $k = \min\{j : c_j \ne 0\}$. From $\sum_{j=k}^{m} c_j \eta^{(j)} = \exp(-i\lambda t)t^N \phi$ we obtain $\sum_{j=k}^{m} c_j \eta^{(j-k)} = P^k(\exp(-i\lambda t)t^N \phi) + p$ for some polynomial p of degree at most $k-1$. But $\eta^{(j)} \in BC_{w_N}(\mathbb{R}, X)$ for $0 \le j \le m$ and so by Theorem 4.8 again we conclude $P^k(\exp(-i\lambda t)t^N \phi) \in \mathcal{F}$. Since $c_k \ne 0$ we can rearrange the differential equation and obtain $\eta \in \mathcal{F}$. □

Remark 6.3. *The asymptotic behaviour of bounded solutions of equations more general than (5.1) are investigated by numerous authors (see [2], [3], [6], [8], [13], [26], [27], [30]). In particular, it follows from [12, Theorem 4.7] that if* $\phi \in BUC_w(J, X)$, $sp_{AP_w}(\phi)$ *is countable and* $\gamma^{-1}\phi \in E_w(J, X)$ *for all* $\gamma \in sp_{AP_w}(\phi)$, *then* $\phi \in AP_w(\mathbb{R}, X)$. *In this paper for solutions of (5.1) we have replaced the ergodicity condition by* $X \not\supset c_0$. *This is satisfied, in particular, if* X *is finite dimensional or reflexive or weakly sequentially complete. So, the results of Theorems 4.6, 6.2 are new even for* $X = \mathbb{R}$ *or* $\mathbb{C}$.

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GAUSSIAN UPPER BOUNDS FOR HEAT KERNELS OF A CLASS OF NONDIVERGENCE OPERATORS

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Abstract. Let Ω be a subset of a space of homogeneous type. Let A be the infinitesimal generator of a positive semigroup with Gaussian kernel bounds on $L^2(\Omega)$. We then show Gaussian heat kernel bounds for operators of the type bA where b is a bounded, complex valued function.

1. Introduction

Behaviour of heat kernels has long been an active topic in functional analysis and partial differential equations. In the past few years, it is known that heat kernel bounds such as Gaussian bounds or Poisson bounds imply various useful properties for operators such as L^p spectral invariance [1], [9], bounded holomorphic functional calculi on L^p spaces [11], $L^p - L^q$ maximal regularity for abstract Cauchy problems [12], [7], L^p-analyticity of the semigroup [15]. A large class of divergence form differential operators on the Euclidean space $\mathbb{R}^D$ are known to possess Gaussian heat kernel bounds, see [8], [2], [4] and their references. However, nondivergence operators with L^∞ coefficients, or even with uniformly continuous coefficients, do not possess Gaussian bounds in general, [5], [6]. Hence we can only hope Gaussian heat kernel bounds for specific classes of nondivergence form operators.

The nondivergence form operators $-b\Delta$ on the Euclidean space $\mathbb{R}^D$ were studied in [14]. It was proved that if $b : X \to \mathbb{C}$ is any bounded measurable function on $\mathbb{R}^D$ such that

$$\Re b(x) \geq \delta > 0 \text{ for a.e. } x \in X \tag{1}$$

then the kernel $k_t(x, y)$ of the semigroup e^{-tbA} has an upper bound with polynomial decay in $|x - y|$. It was also observed that one can improve it to exponential decay by controlling the constants in the upper bound.

The proof in [14] depends on the specific Laplacian Δ through certain estimates using Sobolev embedding and contraction property of the heat semigroup $e^{-t\Delta}$ on $\mathbb{R}^D$. Note that since b is complex-valued, the semigroup $e^{-tb\Delta}$ is no longer contractive.

This result was extended in [10] where it was shown that if the heat kernel $p_t(x, y)$ of an operator A has a Gaussian bound

$$0 \leq p_t(x, y) \leq \frac{C}{t^{D/2}} e^{-c\frac{d^2(x,y)}{t}}. \tag{2}$$

then a similar upper bound holds for the heat kernels of bA. The proof in [10] relies on the Trotter product formula and estimates on the resolvents. During a seminar, the second named author was asked by E. B. Davies if the result in [10] was still true in more general setting like manifolds where the heat kernel bounds take the form

$$0 \leq p_t(x, y) \leq \frac{C}{v(x, \sqrt{t})} e^{-c\frac{d^2(x,y)}{t}}.$$

This is a natural question and this paper is to give a positive answer when the underlying space is a space of homogeneous type or a subset of a space of homogeneous type.

Throughout this paper, C, C', c and c' denote positive constants whose value may change from line to line

2. Main result

Let (X, d, μ) denote a metric space equipped with a $\sigma-$finite measure μ. We assume that X satisfies the doubling property

$$v(x, 2r) \leq Mv(x, r) \quad \forall x \in X, \forall r > 0, \tag{3}$$

where M is a constant and $v(x, r)$ denotes the volume of the ball with center x and radius r.
Suppose that $-A$ is the generator of a bounded analytic semigroup e^{-tA} on $L^2(X, \mu)$ which has a kernel $p_t(x, y)$ satisfying a Gaussian upper bound

$$0 \leq p_t(x, y) \leq \frac{C}{v(x, \sqrt{t})} e^{-c\frac{d^2(x,y)}{t}} \tag{4}$$

for some positive constants C, c and for all $t > 0$.

The aim of this paper is to show that if $b : X \to \mathbb{C}$ is any bounded measurable function on X which satisfies condition (1), then e^{-tbA} has a kernel $k_t(x, y)$ which satisfies a similar Gaussian upper bound, that is

$$|k_t(x, y)| \leq \frac{C'}{v(x, \sqrt{t})} e^{-c'\frac{d^2(x,y)}{t}} \tag{5}$$

for some positive constants C, c and for all $t > 0$.

As we mentioned above if v is polynomial in r and independent of the centre x, i.e., $v(x, r) = cr^D$ for all $x \in X$, $r > 0$, condition (4) becomes condition (2) and a similar estimate to (5) (with $v(x, \sqrt{t})$ replaced by

$t^{D/2}$) was shown in [10]. In a slightly more general setting, Gaussian lower bounds of $k_t(x,y)$ were also studied in [16].

The proof of our main result relies on the same strategy as in [10] in the sense that it is based on the estimates of powers of the resolvents and the Trotter product formula for semigroups. However, the local nature of the upper bound $(v(x,\sqrt{t}))^{-1}$ requires different approach from the case of the uniform bound $t^{-D/2}$. Indeed, an uniform bound on the kernel of an operator can be obtained through the cross norm of the operator itself from the space L^1 to the space L^∞ but this approach is clearly not enough to produce the upper bound of the type $(v(x,\sqrt{t}))^{-1}$. We overcome this problem by using the resolvent equation which implies the representation (8), together with careful estimates on their kernels; see estimates (9) and (10).

We also need to know how to pass from the upper bound (4) on heat kernels to estimates on kernels of powers of the resolvents and vice versa. The following theorem gives that equivalence.

Theorem 1. *Suppose that $-A$ is the generator of a semigroup e^{-tA} which is bounded analytic with angle ν on $L^2(X,\mu)$. The following assertions are equivalent*
(α) *e^{-tA} has a kernel $p_t(x,y)$ which satisfies the estimate (5) for all $t>0$*
(β) *For all $\lambda>0$ and large enough integer m, $(\lambda I+A)^{-m}$ has a kernel $R_{\lambda,m}(x,y)$ which satisfies*

$$|R_{\lambda,m}(x,y)| \le \frac{C}{|\lambda|^m v(x,\frac{1}{\sqrt{|\lambda|}})} e^{-c\sqrt{|\lambda|}d(x,y)} \tag{6}$$

(γ) *for all $\theta\in[0,\nu)$, $\lambda\in\Sigma(\theta+\frac{\pi}{2})$ and large enough integer m, $(\lambda I+A)^{-m}$ has a kernel $R_{\lambda,m}(x,y)$ which satisfies (6) where $\Sigma(\alpha)$ denotes the sector $\{z\in\mathbb{C}, |\arg(z)|<\alpha\}$.*

Remark.

(a) The doubling property (3) implies the strong homogeneity property

$$\forall x\in X, r>0, \lambda\ge 1, \quad v(x,\lambda r)\le M\lambda^n v(x,r) \tag{7}$$

for some $n>0$. This property will be used in our proof.

(b) The condition m large enough in the above theorem can be replaced by $m>n$.

(c) The main result of this paper is Theorem 2 but Theorem 1 is also of independent interest.

We will use Theorem 1 and the Trotter product formula to prove the following.

Theorem 2. *Assume that $-A$ generates a bounded analytic semigroup which has a kernel $p_t(x,y)$ satisfying the upper bound (4). Assume that $b \in L^\infty(X,\mu,\mathbb{C})$, satisfies (1) and the operator $-bA$ generates a bounded analytic semigroup e^{-tbA} on $L^2(X,\mu)$. Then e^{-tbA} has a kernel $k_t(x,y)$ which satisfies (5).*

Remarks.

(a) Sufficient conditions in terms of b and A which ensure that $-bA$ generates a bounded analytic semigroup are given in [10]. For example, $-bA$ always generates a bounded analytic semigroup if A is self-adjoint.

(b) In the above theorem we assumed that $p_t(x,y)$ is positive but the result is still true if we replace the assumption (4) by

$$|p_t(x,y)| \le h_t(x,y) \le \frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}} \tag{4'}$$

where $h_t(x,y)$ is a positive heat kernel. In this case, the right hand side of the estimate (14) below will be $\|b^{-1}\|_\infty(\lambda c_0 I + H)^{-1}|f|$, where H denotes the generator of the semigroup whose kernel is $h_t(x,y)$. The rest of the proof needs no change. As an example, we obtain a Gaussian bound for the heat kernel of bA where $A = \sum_{k=1}^D (\frac{\partial}{\partial x_k} - ia_k)^2$ the magnetic Laplacian on $\mathbb{R}^D$. It is well known that the heat kernel of A satisfies (4$'$) with $h_t(x,y)$ the classical heat kernel of the Laplacian, see [13] for example.

We can replace the space X which satisfies the doubling property (3) by Ω where Ω is any subet of X and the result of Theorem 2 is still true. More specifically, the following theorem can be proved by using the same proof as that of Theorem 2.

Theorem 3. *Assume that $-A$ generates a bounded analytic semigroup on $L^2(\Omega)$ which has kernel $p_t(x,y)$ satisfying*

$$0 \le p_t(x,y) \le \frac{C}{v^X(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}} \tag{4''}$$

for all $t > 0$, all $x, y \in \Omega$, where $v^X(x,\sqrt{t})$ denotes the volume of the ball with centre x, radius $\sqrt{t}$ in the space X. Assume that $b \in L^\infty(\Omega,\mu,\mathbb{C})$, satisfies (1) and the operator $-bA$ generates a bounded analytic semigroup e^{-tbA} on $L^2(\Omega)$.

Then e^{-tbA} has a kernel $k_t(x,y)$ which satisfies

$$|k_t(x,y)| \le \frac{C'}{v^X(x,\sqrt{t})} e^{-c'\frac{d^2(x,y)}{t}} \tag{5''}$$

for all $t > 0$, all $x, y \in \Omega$.

Remarks.

In Theorems 2 and 3, we only give heat kernel bounds on $k_t(x,y)$ for $t > 0$. Using Proposition 3.3 of [11], we can extend the results to obtain similar heat kernel bounds on $k_z(x,y)$ for complex z in some sector of the complex plane.

Applications

We give a few applications of our Theorems 2 and 3:

(a) Let A be the Laplace-Beltrami operator on a manifold $\mathbf{M}$ which satisfies a Sobolev inequality and the doubling property. Then A generates a positive semigroup with Gaussian heat kernel bounds (4). It follows from Theorem 2 that the semigroup e^{-tbA} has Gaussian heat kernel bounds (5).

(b) Let A be a divergence form elliptic operator with real, symmetric coefficients acting on a bounded domain Ω of $\mathbb{R}^D$ with Neumann boundary conditions. Assume that the boundary of Ω satisfies the extension property. Then A generates a positive semigroup with Gaussian heat kernel bounds [8]. More specifically,

$$0 \le p_t(x,y) \le C \max\{1, \frac{1}{t^{D/2}}\} e^{-c\frac{d^2(x,y)}{t}} \qquad (4''').$$

It follows from Theorem 3 that the semigroup e^{-tbA} has Gaussian heat kernel bounds

$$|k_t(x,y)| \le C' \max\{1, \frac{1}{t^{D/2}}\} e^{-c'\frac{d^2(x,y)}{t}} \qquad (5''').$$

(c) As a consequence of heat kernel bounds, the two operators bA in applications (a) and (b) above have the following properties:

(i) L^p spectral invariance: The connected components of their resolvent sets which contain the positive real line on L^p spaces are independent of p, $1 \le p \le \infty$,

(ii) The $L^p - L^q$ maximal regularity for abstract Cauchy problems,

(iii) If bA has a bounded holomorphic functional calculus on L^2, then it has a bounded holomorphic functional calculus on L^p, $1 < p < \infty$.

3. The Proofs.

Proof of Theorem 1. We first show $(\alpha) \Rightarrow (\beta)$.

Let $\lambda > 0$. The Laplace transform gives

$$(\lambda I + A)^{-m} = \frac{1}{m!} \int_0^\infty t^{m-1} e^{-\lambda t} e^{-tA} dt.$$

Hence the kernel $R_{\lambda,m}(x,y)$ of $(\lambda I + A)^{-m}$ is given by

$$|R_{\lambda,m}(x,y)| \leq \frac{C}{m!} \int_0^\infty \frac{t^{m-1}e^{-\lambda t}}{v(x,\sqrt{t})} e^{\frac{-cd^2(x,y)}{t}} dt.$$

Using the fact that $\lambda t + \frac{d^2(x,y)}{t} \geq \sqrt{\lambda} d(x,y)$, we have

$$|R_{\lambda,m}(x,y)| \leq \frac{Ce^{-c'\sqrt{\lambda}d(x,y)}}{m!} \left[\int_0^{\lambda^{-1}} \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})} dt + \int_{\lambda^{-1}}^\infty \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})} dt \right].$$

For $t \in [\lambda^{-1}, \infty)$, we obviously have $v(x,\sqrt{t}) \geq v(x,\sqrt{\lambda^{-1}})$. Hence the second term in the square bracket satifies

$$\begin{aligned} \int_{\lambda^{-1}}^\infty \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})} dt &\leq \frac{1}{v(x,\sqrt{\lambda^{-1}})} \int_{\lambda^{-1}}^\infty t^{m-1} e^{-c"\lambda t} dt \\ &= \frac{1}{v(x,\sqrt{\lambda^{-1}})} \int_1^\infty \lambda^{-m} s^{m-1} e^{-c"s} ds \\ &= \frac{C}{\lambda^m v(x,\sqrt{\lambda^{-1}})}. \end{aligned}$$

For the first term of the square bracket, we have

$$\int_0^{\lambda^{-1}} \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})} dt = \frac{1}{\lambda^m} \int_0^1 \frac{s^{m-1}e^{-c"s}}{v(x,\sqrt{\frac{s}{\lambda}})} ds.$$

We now apply the strong homogeneity property (7) to deduce that

$$v(x,\sqrt{\lambda^{-1}}) \leq M s^{\frac{-n}{2}} v(x,\sqrt{\frac{s}{\lambda}}) \quad \forall s \in (0,1].$$

This implies that

$$\int_0^{\lambda^{-1}} \frac{t^{m-1}e^{-c"\lambda t}}{v(x,\sqrt{t})} dt \leq \frac{C}{\lambda^m v(x,\sqrt{\lambda^{-1}})} \int_0^1 s^{m-\frac{n}{2}-1} e^{-c"s} ds$$

and the last integral is finite for $m > \frac{n}{2}$. This shows the desired implication.

We now show $(\beta) \Rightarrow (\gamma)$. Firstly, we show an upper bound for $R_{m,\lambda}(x,y)$ for $\lambda \in \Sigma(\theta + \frac{\pi}{2})$.
By the resolvent equation (iterated $m-$times) we have

$$(\lambda I + A)^{-2m} = (\lambda I + A)^{-m}(I + (|\lambda| - \lambda)(\lambda I + A)^{-1})^{2m}(\lambda I + A)^{-m}$$

from which it follows that

$$R_{\lambda,2m}(x,y) = \int_X R_{|\lambda|,m}(x,z)(LR_{|\lambda|,m}(.,y))(z) d\mu(z) \tag{8}$$

where $L = (I + (|\lambda| - \lambda)(\lambda I + A)^{-1})^{2m}$. In order to apply L to $z \to R_{|\lambda|,m}(z, y)$ we need to know that the latter is in $L^2(X, \mu)$. Let us assume this for the moment.

By Cauchy-Schwarz inequality and (8) we have

$$|R_{\lambda,2m}(x, y)| \leq \|R_{|\lambda|,m}(x, .)\|_2 \ \|LR_{|\lambda|,m}(., y)\|_2.$$

By the analyticity assumption on e^{-tA} we have

$$\sup_{\lambda \in \Sigma(\theta + \frac{\pi}{2})} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{L}(L^2)} < \infty.$$

In particular,

$$\|LR_{|\lambda|,m}(., y)\|_2 \leq M \|R_{|\lambda|,m}(., y)\|_2$$

with some constant M independent of λ. We then obtain

$$|R_{\lambda,2m}(x, y)| \leq \|R_{|\lambda|,m}(x, .)\|_2 \ \|R_{|\lambda|,m}(., y)\|_2 \tag{9}$$

We now estimate $\|R_{|\lambda|,m}(x, .)\|_2$. Using heat kernel bound (4), we have

$$\begin{aligned}
&\int_X | \ R_{|\lambda|,m}(x, z)|^2 d\mu(z) \\
&\quad \leq \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^2} \int_X e^{-c\sqrt{|\lambda|} d(x,z)} d\mu(z) \\
&\quad = \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^2} \sum_{k=0}^{\infty} \int_{\{\frac{k}{\sqrt{|\lambda|}} \leq d(x,z) \leq \frac{k+1}{\sqrt{|\lambda|}}\}} e^{-c\sqrt{|\lambda|} d(x,z)} d\mu(z) \\
&\quad \leq \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^2} \sum_{k=0}^{\infty} v(x, \frac{k+1}{\sqrt{|\lambda|}}) e^{-ck} \\
&\quad \leq \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})} \sum_{k=0}^{\infty} (k+1)^n e^{-ck}
\end{aligned}$$

where we used (7) to obtain the last inequality. We now obtain from this and (9) that

$$|R_{\lambda,2m}(x, y)| \leq \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^{1/2} v(y, \frac{1}{\sqrt{|\lambda|}})^{1/2}}. \tag{10}$$

Now, Proposition 3.3 of [11] shows that the strong homogeneity property (7), together with the bound (10) for $\lambda > 0$, imply that

$$|R_{\lambda,2m}(x, y)| \leq \frac{C}{|\lambda|^{2m} v(x, \frac{1}{\sqrt{|\lambda|}})^{1/2} v(y, \frac{1}{\sqrt{|\lambda|}})^{1/2}} e^{-c" \sqrt{|\lambda|}} d(x, y)$$

for $\lambda \in \Sigma(\theta + \frac{\pi}{2})$. This inequality and (7) imply (6) for $\lambda \in \Sigma(\theta + \frac{\pi}{2})$.

We now show $(\gamma) \Rightarrow (\alpha)$. The proof is standard but we give the details for the sake of completeness.
Using the inverse Laplace transform we have

$$p_t(x,y) = \frac{m-1}{2\pi i t^{m-1}} \int_{\Gamma_R} e^{\lambda t} R_{\lambda,m}(x,y)d\lambda$$

where $\Gamma_R = \{re^{-i\alpha}, r \geq R\} \cup \{Re^{i\alpha}, |\phi| \leq \alpha\} \cup \{re^{i\alpha}, r \geq R\} := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and where $\alpha \in (\frac{\pi}{2}, \nu + \frac{\pi}{2})$ is a given constant and $R = \max(\frac{1}{t}, \frac{d^2(x,y)}{t^2})$.
Using the assertion 3, we can write that for some constants C, c, c' independent of R

$$|p_t(x,y)| \leq \frac{C}{t^{m-1}} \int_{\Gamma_R} \frac{e^{\Re\lambda t}}{|\lambda|^m v(x, \sqrt{|\lambda|^{-1}})} e^{-c\sqrt{|\lambda|}d(x,y)} d|\lambda| \tag{11}$$

The doubling property (7) implies that for $|\lambda| \geq R \geq t^{-1}$,

$$v(x, \sqrt{t}) \leq M(|\lambda|t)^{\frac{n}{2}} v(x, \sqrt{|\lambda|^{-1}}). \tag{12}$$

Hence,

$$\begin{aligned} &\tfrac{1}{t^{m-1}} \int_{\Gamma_1 \cup \Gamma_3} \tfrac{e^{\Re\lambda t}}{|\lambda|^m v(x,\sqrt{|\lambda|^{-1}})} \; e^{-c\sqrt{|\lambda|}d(x,y)} d|\lambda| \\ &\qquad \leq \frac{C}{v(x,\sqrt{t})} \int_R^\infty (\lambda t)^{-m+\frac{n}{2}} e^{-c'\lambda t} e^{-c\sqrt{\lambda}d(x,y)} d\lambda \\ &\qquad \leq \frac{C}{v(x,\sqrt{t})} e^{-c\sqrt{R}d(x,y)} e^{-\frac{c'}{2}tR} \int_1^\infty s^{-m+\frac{n}{2}} e^{-\frac{c'}{2}s} ds \\ &\qquad \leq \frac{C}{v(x,\sqrt{t})} e^{-c\sqrt{R}d(x,y)} e^{-\frac{c'}{2}tR} \ldots \end{aligned}$$

Using the fact that $R = \max(\frac{1}{t}, \frac{d^2(x,y)}{t^2})$, the last term is dominated by

$$\frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}}.$$

Again by (12) we can bound the third term (i.e., $\int_{\Gamma_2}$) in the right hand side of (11) by

$$\frac{C}{v(x,\sqrt{t})} \int_{|\lambda|=R} (Rt)^{-m+\frac{n}{2}} e^{c'Rt} e^{-c\sqrt{R}t} t d|\lambda|.$$

This term is clearly dominated by

$$\frac{C}{v(x,\sqrt{t})} (Rt)^{-m+\frac{n}{2}+1} e^{-c\sqrt{R}t}$$

which gives the desired bound on $p_t(x,y)$. $\diamondsuit$

Proof of Theorem 2. Suppose that $b \in L^\infty(X,\mu,\mathbb{C})$ satisfies (1) and that $-bA$ generates a bounded analytic semigroup on $L^2(X,\mu)$. For any $\lambda > 0$ we write

$$(\lambda I + bA)^{-1} = (\lambda b^{-1} + A)^{-1} b^{-1} \tag{13}$$

Our aim is to prove the following pointwise inequalities which is valid a.e. for all $f \in L^2(X,\mu)$

$$|(\lambda I + bA)^{-1} f| \le \|b^{-1}\|_\infty (\lambda c_0 I + A)^{-1} |f| \tag{14}$$

where $c_0 = \frac{\delta}{\|b\|_\infty^2}$. The proof of (14) was given in [10] but we repeat it here to keep this paper self sufficient.

We first observe that the positivity of $p_t(x,y)$ implies that the resolvent $(\lambda c_0 I + A)^{-1}$ is a positivity preserving operator for $\lambda > 0$. This is also the case for the operators $(sI + \lambda \Re(\frac{1}{b}) + A)^{-1}$ for all $s, \lambda > 0$ as a consequence of the Trotter product formula (see [17])

$$e^{-t(\lambda\Re(\frac{1}{b})+A)} f = \ \lim_{n\to\infty} (e^{-\frac{t}{n}\lambda\Re(\frac{1}{b})} e^{-\frac{t}{n}A})^n f, \quad \forall f \in L^2(X)$$

(in [17], the formula is given for contraction semigroups but it applies in this situation since both semigroups $e^{-t\Re(\frac{1}{b})}$ and e^{-tA} are contractions for the equivalent norm $\|f\|_* := \sup_{t\ge 0} \|e^{-tA}|f|\|_2$).
Note that by this formula we have the pointwise estimate

$$e^{-t(\lambda\Re(\frac{1}{b})+A)} |f| \le e^{-tA}|f|, \quad \forall t > 0, \text{ and } f \in L^2(X)$$

from which it follows that

$$(sI + \lambda\Re(\frac{1}{b}) + A)^{-1} = \int_0^\infty e^{-st} e^{-t(\lambda\Re(\frac{1}{b})+A)} dt$$

exists for all $s, \lambda > 0$.
Using again the Trotter product formula for $e^{-t(\frac{\lambda}{b}+A)}$, we deduce that

$$|e^{-t(\frac{\lambda}{b}+A)} f| \le e^{-t(\lambda\Re(\frac{1}{b})+A)} |f|$$

which implies

$$|(sI + \frac{\lambda}{b} + A)^{-1} b^{-1} f| \le \|b^{-1}\|_\infty (sI + \lambda\Re(\frac{1}{b}) + A)^{-1} |f|.$$

Since $\Re(\frac{1}{b}) \ge \frac{\delta}{\|b\|_\infty^2} := c_0$, it follows from the Trotter product formula that

$$|(sI + \frac{\lambda}{b} + A)^{-1} b^{-1} f| \le \|b^{-1}\|_\infty (sI + \lambda c_0 I + A)^{-1} |f|.$$

Since $(\frac{\lambda}{b} + A)^{-1}$ and $(\lambda c_0 I + A)^{-1}$ exist we conclude from this inequality that

$$|(\frac{\lambda}{b} + A)^{-1} b^{-1} f| \le \|b^{-1}\|_\infty (\lambda c_0 I + A)^{-1} |f|$$

which is the desired inequality [14].

We now choose an integer $m > n$ where n is the constant in the homogeneity property (7). Iterate (13) $m-$times, we obtain

$$|(\lambda I + bA)^{-m} f| \leq \|b^{-1}\|_\infty^m (\lambda c_0 I + A)^{-m} |f| \ . \tag{15}$$

By assumption, $p_t(x, y)$ satisfies (4) hence by Theorem 1, $(\lambda c_0 I + A)^{-m}$ has a kernel $R_{\lambda,m}(x, y)$ which satisfies (6). In particular, $v(., \sqrt{\lambda^{-1}}) \times (\lambda c_0 I + A)^{-m}$ is bounded from L^1 into L^∞. Estimate (15) implies that $v(., \sqrt{\lambda^{-1}})(\lambda I + bA)^{-m}$ is bounded from L^1 into L^∞, so it is given by a kernel. This implies that $(\lambda I + bA)^{-m}$ is given by a kernel. Again by (15), this kernel satisfies (6). By Theorem 1 we conclude that the kernel $k_t(x, y)$ of e^{-tbA} satisfies (5). $\diamondsuit$

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WEAK TYPE (1,1) ESTIMATES OF MAXIMAL TRUNCATED SINGULAR OPERATORS

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Abstract. Let $\mathcal{X}$ be a space of homogeneous type and T a singular integral operator which is bounded on $L^2(\mathcal{X})$. We give a sufficient condition on the kernel of T so that the maximal truncated operator T_*, which is defined by $T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$, to be of weak type $(1,1)$. Our condition is weaker than the usual Hörmander type condition. Applications include the dominated convergence theorem of holomorphic functional calculi of linear elliptic operators on irregular domains.

1. Introduction and main theorem

Let us consider a space of homogeneous type $(\mathcal{X}, d, \mu)$ which is a set $\mathcal{X}$ endowed with a distance d and a non-negative Borel measure μ on $\mathcal{X}$ such that the doubling condition

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty$$

holds for all $x \in \mathcal{X}$ and $r > 0$, where $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$. A more general defintion can be found in [CW, Chapter 3].

The doubling property implies the following strong homogeneity property,

$$\mu(B(x; \lambda r)) \leq c\lambda^n \mu(B(x; r)) \tag{1.1}$$

for some $c, n > 0$ uniformly for all $\lambda \geq 1$. The parameter n is a measure of the dimension of the space. There also exist c and $N, 0 \leq N \leq n$ so that

$$\mu(B(y; r)) \leq c\left(1 + \frac{d(x, y)}{r}\right)^N \mu(B(x; r)) \tag{1.2}$$

uniformly for all $x, y \in \mathcal{X}$ and $r > 0$. Indeed, the property (1.2) with $N = n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the case of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

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We consider the following "generalised approximations to the identity" which was introduced in [DM].

DEFINITION 1.1. *A family of operators* $\{A_t, t > 0\}$ *is said to be a "generalised approximation to the identity" if, for every* $t > 0$, A_t *is represented by kernels* $a_t(x, y)$ *in the following sense: for every function* $f \in L^p(\mathcal{X}), p \geq 1$,

$$A_t f(x) = \int_{\mathcal{X}} a_t(x, y) f(y) d\mu(y);$$

and the following condition holds:

$$|a_t(x, y)| \leq h_t(x, y) = \frac{1}{\mu(B(x; t^{1/m}))} s(d(x, y)^m t^{-1}), \tag{1.3}$$

where m *is a positive fixed constant and* s *is a positive, bounded, decreasing function satisfying*

$$\lim_{r \to \infty} r^{n+N+\epsilon} s(r^m) = 0 \tag{1.4}$$

for some $\epsilon > 0$, *where* n *and* N *are two constants in (1.1) and (1.2).*

The operators we are going to consider henceforth were introduced in [DM]. They are defined in the following way.

(1.5) T is a bounded operator on $L^2(\mathcal{X})$ with an associated kernel $k(x, y)$ such that for $f \in L_0^\infty(\mathcal{X})$,

$$T(f)(x) = \int_{\mathcal{X}} k(x, y) f(y) d\mu(y), \quad \text{for } \mu - \text{almost every } x \notin \text{supp} f.$$

(1.6) There exists a "generalised approximation of the identity" $\{A_t, t > 0\}$ such that TA_t have associated kernels $k_t(x, y)$ and there exist constants $c_1, c_2 > 0$ so that

$$\int_{d(x,y) \geq c_1 t^{1/m}} |k(x, y) - k_t(x, y)| d\mu(x) \leq c_2, \quad \text{for all } y \in \mathcal{X}.$$

(1.7) There exists a "generalised approximation of the identity" $\{B_t, t > 0\}$ such that $B_t T$ have kernels $K_t(x, y)$ which satisfy

$$|K_t(x, y)| \leq c_4 \frac{1}{\mu(B(x; t^{1/m}))}, \quad \text{when } \ d(x, y) \leq c_3 t^{1/m}$$

and

$$|K_t(x, y) - k(x, y)| \leq c_4 \frac{1}{\mu(B(x; d(x, y)))} \frac{t^{\alpha/m}}{d(x, y)^\alpha}, \text{ when } d(x, y) \geq c_3 t^{1/m},$$

for some constants $c_3, c_4, \alpha > 0$.

We assume that T is an operator satisfying (1.5), (1.6) and (1.7). The maximal operator T_* is the supremum of the truncated integrals, namely,

$$T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)| = \sup_{\epsilon>0} \left| \int_{d(x,y)\geq\epsilon} k(x,y)f(y)d\mu(y) \right|.$$

It is proved in [DM] that if T verifies (1.5) and (1.6), then it is of weak type (1,1) and of strong type (p,p) for $1 < p \leq 2$. In addition, if (1.7) is also satisfied, the operator T is bounded on $L^p(\mathcal{X})$ for all $1 < p < \infty$. Furthermore, Theorem 3 [DM] shows that T_* is bounded on $L^p(\mathcal{X}), 1 < p < \infty$. Implicitly in the proof we can find the following Cotlar type inequality:

$$T_*f(x) \leq CM(Tf)(x) + CMf(x),$$

where M is the Hardy-Littlewood maximal function. Hence, boundedness of T_* follows from boundedness of T and M.

The following is the main result of this paper.

THEOREM 1.2. *Let T be an operator satisfying the assumptions (1.5) and (1.7). Also assume the following condition (1.8): there exists a "generalised approximation of the identity" $\{A_t, t > 0\}$ so that the kernels $(\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y))$ of the operators $(B_\epsilon T A_t - B_\epsilon T)$ satisfy*

$$\sup_\epsilon \int_{d(x,y)\geq\beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y)| d\mu(x) \leq C \tag{1.8}$$

for some constants C and β, and for all $y \in \mathcal{X}$. Then,

(i) the maximal truncated operator T_* is bounded on $L^p(\mathcal{X})$ for $1 < p < \infty$.

(ii) When $p = 1$, T_* is of weak-type $(1,1)$, that is,

$$\mu(\{x : |T_*f(x) > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1, \quad \text{for all } \alpha > 0.$$

NOTE: (i) Comparing the above Theorem 1.2 with Theorem 3 of [DM], the assumption (1.8) is stronger than (1.6), but we obtain new endpoint estimates, i.e. the weak type (1,1) estimates in (ii).

(ii) Theorem 1.2 improves the classical results of Calderón-Zygmund operators. See [St, Chapter 1, Corollary 2] for Euclidean spaces $\mathcal{X} = \mathbb{R}^n$, and [CW] for spaces of homogeneous type. Let us note that there is no regularity assumptions in the space variables. In comparison with the classical Calderón-Zygmund operators, the Hörmander type inequalities are replaced by (1.7) and (1.8) which involved the "generalised approximations of the identity". In fact, for suitable "generalised approximations of the identity", it is proved in [DM] that conditions (1.6) and (1.7) are weaker than the usual assumptions of Calderón-Zygmund operators (Proposition 2, [DM]). We also show that our condition (1.8) is actually a consequence of the condition (1.6) (Proposition 2.1). As applications, we get the dominated convergence theorem of holomorphic functional calculi of linear elliptic operators on irregular domains.

2. Proof of Theorem 1.2

We first prove (ii). For a fixed $\epsilon > 0$, one writes $T_\epsilon u(x) = B_{\epsilon^m} Tu(x) - (B_{\epsilon^m} T - T_\epsilon)(u)(x)$. Since the class of operators B_t satisfies the conditions (1.3) and (1.4), we have

$$|B_{\epsilon^m} Tu(x)| \leq cM(|Tu(x)|) \tag{2.1}$$

where c is a constant independent of ϵ. Similarly to the proof of Theorem 3 in [DM], using the condition (1.7) we also have

$$\sup_{\epsilon>0} |(B_{\epsilon^m} T - T_\epsilon)u(x)| \leq cM(|u|(x)). \tag{2.2}$$

Theorem 1.2 then follows from (2.2) if we can prove that the operator

$$T_*^B u(x) = \sup_{\epsilon} |B_{\epsilon^m} Tu(x)|$$

is of weak-type $(1,1)$. Following the idea of Theorem 1 of [DM], we first use the Calderón-Zygmund decomposition to decompose an integrable function into "good" and "bad" parts (see, for example, [CW]).

Given $f \in L^1(\mathcal{X}) \cap L^2(\mathcal{X})$ and $\alpha > \|f\|_1 (\mu(\mathcal{X}))^{-1}$, then there exist a constant c independent of f and α, and a decomposition

$$f = g + b = g + \sum_i b_i,$$

so that

(a) $|g(x)| \leq c\alpha$ for all almost $x \in \mathcal{X}$;

(b) there exists a sequence of balls Q_i so that the support of each b_i is contained in Q_i and

$$\int_{\mathcal{X}} |b_i(x)| d\mu(x) \leq c\alpha\mu(Q_i);$$

(c) $\sum_i \mu(Q_i) \leq c\alpha^{-1} \int_{\mathcal{X}} |f| d\mu(x)$;

(d) each point of $\mathcal{X}$ is contained in at most a finite number N of the balls Q_i.

Note that conditions (b) and (c) imply that $\|b\|_1 \leq c\|f\|_1$ and hence that $\|g\|_1 \leq (1 + c)\|f\|_1$.

We have

$$\mu(\{x:\ |T_*^B f(x)|>\alpha\}) \\ \le \mu(\{x:\ |T_*^B g(x)|>\alpha/2\})+\mu(\{x:\ |T_*^B b(x)|>\alpha/2\}).$$

It follows from (2.1), (1.5) and boundedness of the Hardy-Littlewood maximal function that T_*^B is bounded on $L^2(\mathcal{X})$. Since $|g(x)|\le c\alpha$, we obtain

(2.3)
$$\mu(\{x:\ |T_*^B g(x)|>\alpha/2\})\le 4\alpha^{-2}\|T_*^B g\|_2^2\le c\alpha^{-2}\|g\|_2^2\le c\alpha^{-1}\|f\|_1.$$

Concerning the "bad" part $b(x)$, we temporarily fix a b_i whose support is contained in Q_i, then choose $t_i=r_i^m$ where m is the constant appearing in (1.3), and r_i is the radius of the ball Q_i. We then decompose

$$\sup_\epsilon\Big|B_\epsilon T\sum_i b_i(x)\Big| \\ \le \sup_\epsilon\bigg(\Big|B_\epsilon T\sum_i A_{t_i}b_i(x)\Big|+\Big|B_\epsilon T\sum_i(I-A_{t_i})b_i(x)\Big|\bigg).$$

It follows from the decay assumption (1.3) that

$$\Big\|\sum_i A_{t_i}b_i\Big\|_2\le c\alpha\Big(\sum_i\mu(Q_i)\Big)^{1/2}\le c\alpha^{1/2}\|f\|_1^{1/2}.$$

See details in the proof of estimate (10) in [DM]. Combining this with L^2-boundedness of T_*^B, we have

$$\text{(2.4)}\quad \begin{aligned}\mu(\{x:\ \sup_\epsilon|B_\epsilon T\textstyle\sum_i A_{t_i}b_i(x)|>\alpha/4\}) &\le 16\alpha^{-2}\Big\|\displaystyle\sum_i T_*^B A_{t_i}b_i\Big\|_2^2\\ &\le c\alpha^{-2}\Big\|\sum_i A_{t_i}b_i\Big\|_2^2\\ &\le \frac{c}{\alpha}\|f\|_1.\end{aligned}$$

On the other hand

$$\mu(\{x:\ \sup_\epsilon|B_\epsilon T\textstyle\sum_i(I-A_{t_i})b_i(x)|>\alpha/4\}) \\ \le \displaystyle\sum_i\mu(B_i)+\sum_i\frac{4}{\alpha}\int_{(B_i)^c}\sup_\epsilon|B_\epsilon T(I-A_{t_i})b_i(x)|d\mu(x),$$

where $(B_i)^c$ denotes the complement of B_i which is the ball with the same centre y_i as that of the ball Q_i in the Calderón-Zygmund decomposition but with radius increased by a factor of $(1+c_1)$, where c_1 is the constant in (1.6). Because of property (c) of the Calderón-Zygmund decomposition and doubling volume property of $\mathcal{X}$, we have

$$\sum_i \mu(B_i) \le c \sum_i \mu(Q_i) \le c\alpha^{-1}\|f\|_1. \tag{2.5}$$

Using assumption (1.8), we have

$$\begin{aligned}
&\int_{(B_i)^c} \sup_\epsilon |B_\epsilon T(I - A_{t_i})b_i(x)| d\mu(x) \\
&\le \int_{(B_i)^c} \sup_\epsilon \left| \int_{\mathcal{X}} (K_\epsilon(x,y) - \mathcal{K}_{\epsilon,t_i}(x,y)) b_i(y) d\mu(y) \right| d\mu(x) \\
&\le \int_{\mathcal{X}} \|b_i(y)\| \left\{ \sup_y \sup_\epsilon \int_{d(x,y)\ge c{t_i}^{1/m}} |K_\epsilon(x,y) - \mathcal{K}_{\epsilon,t_i}(x,y)| d\mu(x) \right\} d\mu(y) \\
&\le C\|b_i\|_1, \qquad \text{because } B(y; ct_i^{1/m}) \subset B_i.
\end{aligned}$$

Therefore

$$\sum_i \frac{1}{\alpha} \int_{(B_i)^c} \sup_\epsilon |B_\epsilon T(I - A_{t_i})b_i(x)| d\mu(x) \le C\alpha^{-1} \sum_i \|b_i\|_1 \le \frac{C}{\alpha}\|f\|_1. \tag{2.6}$$

Combining the above estimates (2.2), (2.3), (2.4), (2.5) and (2.6), we have for any $\alpha > \|f\|_1(\mu(\mathcal{X}))^{-1}$,

$$\mu(\{x : \ |T_* f(x)| > \alpha\}) \le \frac{C}{\alpha}\|f\|_1.$$

If $\mathcal{X}$ is unbounded, the proof is done because the former inequality holds for every $\alpha > 0$. Otherwise, we have to consider what happens for $0 < \alpha \le \|f\|_1(\mu(\mathcal{X}))^{-1}$. Since $\mathcal{X}$ is bounded we can write $\mathcal{X} = B(x_0, r)$ for some $r > 0$. We conclude

$$\mu(\{x : |T_* f(x)| > \alpha\}) \le \mu(\mathcal{X}) \le \frac{C}{\alpha}\|f\|_1$$

for any $\alpha > 0$.

We now prove (i). For any $1 < p \le 2$, L^p-boundedness of T_* follows from the Marcinkiewicz interpolation theorem. Using a standard duality argument, T_* is proved to be a bounded operator on $L^p(\mathcal{X})$ for all $2 < p < \infty$.

The proof of Theorem 1.2 is complete.

In the next proposition, we show that, for suitable chosen B_t, our condition (1.8) is actually a consequence of condition (1.6).

PROPOSITION 2.1. *Let T be a bounded linear operator on $L^2(\mathcal{X})$ with kernel $k(x,y)$. Assume there exists a "generalised approximation of* *the identity" $\{A_t, t>0\}$ so that the kernels $k_t(x,y)$ of TA_t satisfy the condition (1.6), i.e. there exist constant c and $\delta>1$ so that*

$$\int_{d(x,y)\geq\delta t^{1/m}} |k(x,y)-k_t(x,y)|d\mu(x)\leq c \tag{2.7}$$

for all $y\in\mathcal{X}$.

Then, there exists a "generalised approximation of the identity" $\{B_t, t>0\}$ which is represented by kernels $b_t(x,y)$ in the following sense: for any $f\in L^p(\mathcal{X}), p\geq 1$,

$$B_tf(x)=\int_{\mathcal{X}} b_t(x,y)f(y)d\mu(y),$$

so that the kernels $(\mathcal{K}_{\epsilon,t}(x,y)-K_\epsilon(x,y))$ of the operators $(B_\epsilon TA_t-B_\epsilon T)$ satisfy

$$\sup_\epsilon \int_{d(x,y)\geq\beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y)-K_\epsilon(x,y)|d\mu(x)\leq C$$

for some constants C and β, and for all $y\in\mathcal{X}$.

Proof. Choose $\delta>1$ and let $\beta=3\delta/2$. For any $\epsilon>0$, we choose $b_\epsilon(x,z)=0$ when $d(x,z)\geq(\delta/2)t^{1/m}$. Then, for $x,y\in\mathcal{X}$ so that

$$\mathcal{K}_{\epsilon,t}(x,y)=\int_{\mathcal{X}} b_\epsilon(x,z)k_t(z,y)d\mu(z),$$

$$\text{and}\qquad K_\epsilon(x,y)=\int_{\mathcal{X}} b_\epsilon(x,z)k(z,y)d\mu(z).$$

For all $y\in\mathcal{X}$,

$$\begin{aligned}
&\int_{d(x,y)\geq\beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y)-K_\epsilon(x,y)|d\mu(x)\\
&\quad\leq \int_{d(x,y)\geq\beta t^{1/m}}\int_{\mathcal{X}} |b_\epsilon(x,z)||k_t(z,y)-k(z,y)|d\mu(x)d\mu(z)\\
&\quad\leq \left(\sup_{z\in\mathcal{X}}\int_{\mathcal{X}}|b_\epsilon(x,z)|d\mu(x)\right)\times\int_{d(z,y)\geq\delta t^{1/m}}|k_t(z,y)-k(z,y)|d\mu(z)\\
&\quad\leq c_1\int_{d(z,y)\geq\delta t^{1/m}}|k_t(z,y)-k(z,y)|d\mu(z)\\
&\quad\leq C,
\end{aligned}$$

where the last inequality follows from (2.7) and the third inequality is using the estimate

$$\int_{\mathcal{X}}|b_\epsilon(x,z)|d\mu(x)\leq\int_{\mathcal{X}}h_\epsilon(x,z)d\mu(z)\leq c_1.$$

As a consequence of the boundedness of the maximal truncated operator T_*, we obtain pointwise almost everywhere convergence of $\lim_{\epsilon\to 0} T_\epsilon f(x)$. More precisely, we have the following corollary.

COROLLARY 2.2. *Assume that the operator T satisfies the conditions of Theorem 1.2. Assume that the kernel $k(x,y)$ of T satisfies the estimate*

$$|k(x,y)| \leq c(\mu(B(x;d(x,y)))^{-1}.$$

Then there exist a sequence of positive functions $\epsilon_j(x)$ such that $\lim\limits_{j\to\infty} \epsilon_j(x) = 0$, and a function $m \in L^\infty(\mathcal{X})$ such that for $f \in L^p(\mathcal{X}), 1 \leq p < \infty$,

$$Tf(x) = m(x)f(x) + \lim_{j\to\infty} \int_{|x-y|\geq \epsilon_j(x)} k(x,y)f(y)d\mu(y)$$

for almost every $x \in \mathcal{X}$.

Proof. Corollary 2.2 follows from a standard argument of proving the existence of almost everywhere pointwise limits as a consequence of the corresponding maximal inequality. See, for example, [CM, Chapter 7, Theorem 6] for Euclidean spaces $\mathcal{X} = \mathbb{R}^n$, and [CW] for spaces of homogeneous type.

REMARK 2.3.

As in Section 3 of [DM], Theorem 1.2 and Corollary 2.2 can be modified so that they are still true when the space of homogeneous type $\mathcal{X}$ is replaced by one of its measurable subsets Ω. In this sense, it is sufficient that condition (1.3) on the upper bound $h_t(x,y)$ of the kernel $a_t(x,y)$ is replaced by

$$h_t(x,y) = (\mu(B^{\mathcal{X}}(x;t^{1/m})))^{-1} s(d(x,y)^m t^{-1}),$$

where $B^{\mathcal{X}}(x;t^{1/m})$ is the ball of centre x, radius $t^{1/m}$ in the space $\mathcal{X}$. For the details, see Section 3, [DM].

3. Applications: Holomorphic functional calculi of linear elliptic operators

We first review some definitions regarding the holomorphic functional calculus as introduced by McIntosh [Mc]. Let $0 \leq \omega < \pi$ be given. Then

$$S_\omega = \{z \in \mathbb{C} : |\arg z| \leq \omega\} \cup \{0\}$$

denotes the closed sector of angle ω and S_ω^0 denotes its interior, while $\dot{S}_\omega = S_\omega \backslash \{0\}$. An operator L on some Banach space E is said to be of type ω if L is closed and densely defined, $\sigma(L) \subset S_\omega$, and for each $\theta \in (\omega, \pi]$ there exists a constant C_θ such that

$$|\eta| \, \|(\eta I - L)^{-1}\|_{\mathcal{L}(E)} \le C_\theta, \qquad \eta \in -\dot{S}_{\pi-\theta}.$$

If $\mu \in (0, \pi]$, then

$$H_\infty(S^0_\mu) = \{f : S^0_\mu \to \mathbb{C}; f \text{ is holomorphic and } ||f||_\infty < \infty\},$$

where $||f||_{H_\infty} = \sup\{|f(z)| : z \in S^0_\mu\}$. In addition, we define

$$\Psi(S^0_\mu) = \left\{ g \in H_\infty(S^0_\mu) : \exists s > 0, \exists c \ge 0 : |g(z)| \le c \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

If L is of type ω and $g \in \Psi(S^0_\mu)$, we define $g(L) \in L(E)$ by

$$g(L) = -\frac{1}{2\pi i} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta, \tag{3.1}$$

where Γ is the contour $\{\xi = re^{\pm i\theta} : r \ge 0\}$ parametrised clockwise around S_ω, and $\omega < \theta < \mu$. If, in addition, L is one-one and has dense range and if $g \in H_\infty(S^0_\mu)$, then

$$f(L) = [h(L)]^{-1}(fh)(L), \tag{3.2}$$

where $h(z) = z(1+z)^{-2}$. It can be shown that $g(L)$ is a well-defined linear operator in E and that this definition is consistent with definition (3.2) for $g \in \Psi(S^0_\mu)$. The definition of $g(L)$ can even extended to encompass unbounded holomorphic functions; see [Mc] for details. L is said to have a bounded holomorphic functional calculus on the sector S_μ if

$$||g(L)||_{\mathcal{L}(E)} \le N||g||_\infty$$

for some $N > 0$, and for all $g \in H_\infty(S^0_\mu)$.

Assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$. Let L be a linear operator on $L^2(\Omega)$ with $\omega < \pi/2$ so that $(-L)$ generates a holomorphic semigroup $e^{-zL}, 0 \le |\mathrm{Arg}(z)| < \pi/2 - \omega$ which possesses the following two properties:

(3.3) The holomorphic semigroup $e^{-zL}, |\mathrm{Arg}(z)| < \pi/2 - \omega$ is represented by kernels $a_z(x, y)$ which satisfy, for all $\theta > \omega$, an upper bound

$$|a_z(x, y)| \le c_\theta h_{|z|}(x, y)$$

for $x, y \in \Omega$, and $|\mathrm{Arg}(z)| < \pi/2 - \theta$, where $h_{|z|}$ is defined on $\mathcal{X} \times \mathcal{X}$ by (1.3).

(3.4) The operator L has a bounded holomorphic functional calculus in $L^2(\Omega)$. That is, for any $\nu > \omega$ and $g \in H_\infty(S^0_\nu)$, the operator $g(L)$ satisfies

$$||g(L)f||_2 \le c_\nu ||g||_\infty \|f\|_2.$$

Applying Theorem 1.2, Corollary 2.2 and Remark 2.3, we have

THEOREM 3.1. *Let L be an operator verifying the assumptions (3.3) and (3.4). Assume that for $g \in H_\infty(S_\nu^0)$, the kernel $k(x,y)$ of $g(L)$ satisfies the estimate*

$$|k(x,y)| \le c(\mu(B(x;d(x,y)))^{-1}, \quad \text{for all } x,y \in \Omega. \tag{3.5}$$

If we denote $T = g(L)$, then

(i) If $1 < p < \infty$, then $\|T_* f\|_p \le C\|g\|_\infty \|f\|_p$.

(ii) If $f \in L^1(\mathcal{X})$, then the map $f \to T_* f$ is weak type $(1,1)$.

(iii) There exist a sequence of positive functions $\epsilon_j(x)$ such that $\lim_{j\to\infty} \epsilon_j(x) = 0$, and a function $m(x) \in L^\infty(\Omega)$ such that for $f(x) \in L^p(\Omega), 1 \le p < \infty$,

$$Tf(x) = m(x)f(x) + \lim_{j\to\infty} \int_{|x-y|\ge\epsilon_j(x)} K(x,y)f(y)d\mu(y)$$

for almost every $x \in \Omega$.

Proof. We follow an idea of Theorem 6 in [DM]. Choose operators $B_t = A_t = e^{-tL}$. As in Theorem 6 of [DM], there exist some constants $c, c_1, \alpha > 0$ such that the kernels $K_t(x,y)$ of $B_t T$ satisfy

$$|K_t(x,y)| \le c\frac{1}{\mu(B^{\mathcal{X}}(x;t^{1/m}))},$$

for all $x,y \in \Omega$ such that when $d(x,y) \le c_1 t^{1/m}$;

$$|K_t(x,y) - k(x,y)| \le c\frac{1}{\mu(B^{\mathcal{X}}(x;d(x,y)))}\frac{t^{\alpha/m}}{d(x,y)^\alpha}$$

for all $x,y \in \Omega$ such that when $d(x,y) \ge c_1 t^{1/m}$.

Using the methods of Theorems 5 and 6 [DM], it is not difficult to check that the kernels $(\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y))$ of the operators $(B_\epsilon T A_t - B_\epsilon T)$ satisfy

$$\sup_\epsilon \int_{d(x,y)\ge\beta t^{1/m}} |\mathcal{K}_{\epsilon,t}(x,y) - K_\epsilon(x,y)|d\mu(x) \le C \tag{3.6}$$

for some constants C and β, and for all $y \in \Omega$.

For the proof of (3.6), we leave it to readers. Then, Theorem 3.1 follows from Remark 2.3.

REMARK 3.2.

The condition (3.5) is satisfied by large classes of linear operators on $\mathbb{R}^n$ or a domain Ω of $\mathbb{R}^n$ without any condition on smoothness of the boundary of Ω. For example, if the function $h_t(x,y)$ of (1.3) is bounded above by the Gaussian bounds

$$ct^{-n/2}\exp\{-\alpha|x-y|^2/t\}$$

for some $\alpha > 0$, or by the Poisson bounds

$$\frac{ct}{(t^2+|x-y|^2)^{(n+1)/2}},$$

then (3.5) is a direct result from straightforward integration.

One example of an operator L which possesses Gaussian bounds on its heat kernel is the Schrödinger operator with potential V, defined by

$$L = -\triangle + V(x),$$

where V is a nonnegative function on $\mathbb{R}^n$. See, Lecture 7 in [ADM]. Another example of an operator L on such a domain, which possesses Gaussian bounds on its heat kernel, is the Laplacian on an open subset of $\mathbb{R}^n$ subject to Dirichlet boundary conditions. More general operators on open domain of $\mathbb{R}^n$ which possess Gaussian bounds can be found in [AE], [DM].

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L^2 BOUNDS FOR NORMAL DERIVATIVES OF DIRICHLET EIGENFUNCTIONS

ANDREW HASSELL AND TERENCE TAO

ABSTRACT. Suppose that M is a compact Riemannian manifold with boundary and u is an L^2-normalized Dirichlet eigenfunction with eigenvalue λ. Let ψ be its normal derivative at the boundary. Scaling considerations lead one to expect that the L^2 norm of ψ will grow as $\lambda^{1/2}$ as $\lambda \to \infty$. We sketch proofs of an upper bound of the form $\|\psi\|_2^2 \le C\lambda$ for any Riemannian manifold, and a lower bound $c\lambda \le \|\psi\|_2^2$ provided that M has no trapped geodesics (see the main Theorem for a precise statement). Here c and C are positive constants that depend on M, but not on λ. Full details will appear in [3].

1. INTRODUCTION

Let M be a smooth compact Riemannian manifold with smooth boundary $\partial M = Y$ (for example, the closure of a smooth bounded domain in Euclidean space). Let $H = -\Delta_M$ be minus the Dirichlet Laplacian on M. As is well known, H has discrete spectrum $0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots \to \infty$.

Let u_j be an L^2-normalized eigenfunction corresponding to λ_j, and let ψ_j be the normal derivative of u_j at the boundary. In this paper we consider the following question: do there exist constants c and C, depending on M but not on j, such that

$$c\lambda_j \le \|\psi_j\|^2_{L^2(Y)} \le C\lambda_j \quad ?$$

This question was posed by Ozawa in [6]; he showed that a weaker version of this statement, obtained by summing over all eigenvalues in $[0, \lambda]$, is true.

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In this paper, we sketch proofs that the upper bound is always true on Riemannian manifolds, while the lower bound holds provided a condition of 'no trapped geodesics' holds. In addition, we give several examples to illustrate the link between failure of the geodesic condition and failure of the lower bound. Full details of proofs are given in [3].

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2. Examples

We begin by considering several examples. These examples show in particular that the lower bound does *not* always hold.

Example 1 — the disc. Let $M = \{x \in \mathbb{R}^2 \mid |x| < a\}$ for some $a > 0$. In this case we have an equality

$$\int_{S_1} \psi_j^2(\theta)\, d\theta = \frac{2\lambda_j}{a}.$$

This follows easily from an identity for eigenfunctions due to Rellich, which we discuss below.

Example 2 — the rectangle. Let $M = [0,a]\times[0,b]$, where $a \le b$. Then it is a simple matter to write down the eigenfunctions, by separating variables. A computation gives

$$\frac{4}{b}\lambda \le \|\psi\|_2^2 \le \frac{4}{a}\lambda,$$

and these bounds are the best possible.

Example 3 — the cylinder. Let $M = [0,\pi] \times S^1_{2\pi}$, the product of an interval of length π with a circle of length 2π. Then eigenfunctions take the form $\sin(mx)e^{in\theta}$. The upper bound

$$\|\psi\|_2^2 \le \frac{4}{\pi}\lambda$$

holds, but by holding m fixed and sending n to infinity, we see that the best lower bound is $O(1)$.

Example 4 — the hemisphere. Let M be the hemisphere

$$M = \{x \in \mathbb{R}^3 \mid |x| = 1,\ x \cdot (0,0,1) \ge 0\}.$$

In this case, the eigenfunctions are given by those spherical harmonics which are odd under reflection in the (x_1, x_2) plane, namely, spherical harmonics

$$u = c_{lm} Y_{lm} = c_{lm} e^{im\phi} P_{lm}(\cos\theta), \quad \lambda = l(l+1),$$

where $-l \le m \le l$ and $l-m$ is odd. (Here, we are using spherical polar coordinates where $(x_1, x_2, x_3) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.) Let us consider the case when $m = l-1$. Then the eigenfunction in this case is $u_l = ce^{i(l-1)\theta}\cos\theta(\sin\theta)^{l-1}$. A short computation gives

$$\|\psi\|_2^2 \sim l^{3/2} \sim \lambda^{3/4}, \quad l \to \infty,$$

for this class of eigenfunctions. Hence there is no nontrivial lower bound for the hemisphere.

3. Main theorem

The upper bound holds in all the examples in the previous section, but the lower bound fails for the last two. To see more heuristically why the lower bound fails, it helps to consider a more dynamic picture, by considering the wave equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} = -Hv$$

on the cylinder (Example 3). If u is an eigenfunction with eigenvalue λ, then $v = e^{i\sqrt{\lambda}t}u$ is a solution to the wave equation. Thus in the case of the cylinder, a particular solution to the wave equation is

$$2i\sin(mx)e^{in\theta}e^{i\sqrt{m^2+n^2}t} = e^{imx}e^{in\theta}e^{i\sqrt{m^2+n^2}t} - e^{-imx}e^{in\theta}e^{i\sqrt{m^2+n^2}t}.$$

The wavefronts are at $\pm mx + n\theta =$ constant, and energy moves 'normal' to the wavefronts. When we hold m fixed and send n to infinity, the energy is moving more and more along lines (that is, geodesics) where x is constant, and so does not 'reach' the boundary. Thus, the failure seems to be related to the existence of a family of 'trapped' geodesics in the interior of the manifold, which never reach the boundary.

In the case of the hemisphere, there is no geodesic trapped in the interior, but any Riemannian extension N of M would have a trapped geodesic, namely the boundary of M. Note that the lower bound is not violated as severely for the hemisphere, which is consistent with it being a borderline case.

Our main result is that the heuristic above is correct:

Theorem **3.1.** *Let M be a smooth compact Riemannian manifold with boundary. Then the upper bound holds for some C independent of j.*

The lower bound holds provided that M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. In particular, the lower bound holds if M is a subdomain of Euclidean space.

In Section 7, we give some further examples which show that the degree of failure of the bounds is related to the extent to which geodesics are trapped.

4. Upper bound

Our proof is based on the following Lemma which we call a Rellich-type estimate.

Lemma **4.1.** *Let u be a Dirichlet eigenfunction of H. Then for any differential operator A,*

$$\int_M \langle u, [H, A]u \rangle \, dg = \int_Y \frac{\partial u}{\partial \nu} Au \, d\sigma. \tag{4.1}$$

Proof. Let λ be the eigenvalue corresponding to u. We write $[H, A] = [H - \lambda, A]$ and use the fact that $(H - \lambda)u = 0$ to write the integral over M as

$$\int_M \langle (H - \lambda)u, Au \rangle - \langle u, (H - \lambda)Au \rangle \, dg.$$

Then we use Green's formula, and the fact that u vanishes at the boundary, to deduce (4.1). □

The upper bound is now easily deduced. Let us choose coordinates (r, y) near the boundary of M, where r is distance to the boundary (which is a smooth function for $r < \delta$, for some sufficiently small $\delta > 0$) and y are local coordinates on $Y = \partial M$, extended to be constant along geodesics perpendicular to the boundary. Let $\chi(r)$ be a smooth function which is supported in $[0, \delta/2]$ and with $\chi(0) = 1$, and let

$$A = \chi(r)\partial_r.$$

Then the right hand side of (4.1) becomes $\|\psi\|_2^2$, while the left hand side is bounded by

$$C \int_M \big(|\nabla u|^2 + 1 \big) = C(\lambda + 1),$$

proving the upper bound.

Remark. Notice that this argument actually gives a bound of

$$C_\epsilon \|Hu\|_{L^2(\{r \le \epsilon\})} = C_\epsilon \lambda \|u\|_{L^2(\{r \le \epsilon\})}$$

for any $\epsilon > 0$.

5. Lower bound for Euclidean domains

In the case of subdomains of Euclidean space, there is a very simple proof of the lower bound based on Lemma 4.1. We set

$$A = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Then $[H, A] = 2H$, so now the left hand side is 2λ, and we obtain the identity

$$2\lambda = \int_Y \frac{\partial u}{\partial \nu} Au \, d\sigma = \int_Y \nu \cdot x \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma$$

which immediately implies the lower bound. (Rellich published this identity in 1940 [7].)

6. The lower bound in general

It is considerably harder to prove the lower bound for general manifolds satisfying the 'no trapped geodesic' condition of Theorem 3.1. We use the method for Euclidean domains as a guide, and seek a first order operator A having a positive commutator with H. It is impossible to find a vector field A with this property, in general (we will show this later), so we look for a first order pseudodifferential operator. A is essentially determined by its principal symbol, a, a function on S^*M, the cosphere bundle of M.

The principal symbol of $i[H, A]$ is given by $V_h(a)$, where V_h is the Hamilton vector field of the symbol of H. Recall that H is minus the Laplacian. It is well known that V_h is the generator of geodesic flow on S^*M. We want $i[H, A]$ positive, which amounts to finding a smooth function a on S^*M which is *increasing along all geodesics.* Note that this is clearly impossible if trapped/periodic geodesics are present.

Given the 'no trapped geodesics' condition on M in the theorem, such an A can easily be constructed. To show this, we first observe that, given a geodesic γ on M, one can construct an A on the larger manifold $N \supset M$, properly supported in the interior of N, such that the principal symbol of $i[H, A]$ is everywhere nonnegative, and strictly positive in a neighbourhood of γ (at least that part of it lying over M). To do *this* we simply define the symbol a of A to be linearly increasing along γ, extend it in a natural way, and then cut off. Finally, a compactness argument shows that one can add together finitely many such operators to produce one where the principal symbol of $i[H, A]$ is strictly positive on S^*M. We note that it is easy to arrange that, in addition, A satisfies the *transmission condition* (see [5], section 18.2)); for the principal symbol a of A, this condition is simply that a is odd: $a(x, -\xi) = -a(x, \xi)$.

We then follow the strategy of the previous proof. First, we need a version of Lemma 4.1 which is valid for pseudodifferential operators. For first order pseudodifferential operators A satisfying the transmission condition, with symbol a, we have the following

LEMMA **6.1.** *Let u be a Dirichlet eigenfunction for H. Then*

$$\int\limits_M \langle u, [H, A]u\rangle \, dg = 2 \ \mathrm{Im} \int\limits_Y \frac{\partial u}{\partial \nu} Au \, d\sigma - \int\limits_Y \big(\frac{\partial u}{\partial \nu}\big)^2 c \, d\sigma, \tag{6.1}$$

where $c(y) = \lim_{\rho \to \infty} \rho^{-1} a(0, y, \rho, 0)$.

This is just as good as (4.1) for our purposes. We then follow the strategy of the previous proof. The left hand side causes few problems; it is rather easy to show that the left hand side is at least as big as a constant times λ. The right hand side, though, is more difficult. It is sufficient to show that

$$\|Au\|_{L^2(Y)} \leq C\sqrt{\lambda}, \tag{6.2}$$

since then we can estimate the first term on the right hand side of (6.1) by

$$\int_Y \langle \psi, Au\rangle d\sigma \leq C(\epsilon)\|\psi\|_2^2 + \epsilon\lambda, \tag{6.3}$$

and the $\epsilon\lambda$ term may then be taken to the left hand side. The estimate (6.2) is nontrivial, since A is a nonlocal operator; the restriction of Au to the boundary depends on values of u in the interior of M.

Our proof of (6.2) uses ideas from both harmonic analysis and microlocal analysis. There are two main ingredients. One is a uniform estimate for u near the boundary:

$$\int_{Y_r} u_j^2 \, d\sigma(y) \leq C\lambda_j r^2 \text{ for all } r \in [0, \delta] \tag{6.4}$$

where Y_r is the set of points at distance r from the boundary, and both C and δ are *independent of* j. We prove this by looking at the quantity

$$L(r) = \int_{Y_r} u^2 d\sigma_r.$$

Here $d\sigma_r$ is the measure on Y_r induced by the Riemannian measure on M. The eigenfunction equation for u leads to the differential inequality

$$L'' \geq \frac{(L')^2}{L} - C\lambda, \tag{6.5}$$

for $L(r)$, for some constant C depending only on the manifold M. This differential inequality implies exponential increase of $L(r)$ if it ever happens that $L'(r)^2 \geq 2C\lambda L(r)$. However, this would contradict the bound

$$\int_0^\delta L(r)dr \leq 1$$

which follows from the L^2 normalization of u. Hence $L'(r)^2 \leq 2C\lambda L(r)$, and from this we deduce (6.4).

The second ingredient is expressing $Au_{|Y}$ as an integral of kernels A_r acting on the functions $u_{|Y_r}$. Here, A_s has kernel $A_s(y, y')$ which is the restriction of the kernel of A to $(r = 0, y, r' = s, y')$. Then results of Boutet de Monvel [1] and Višik and Eškin [8] give bounds on the L^2 operator norm of A_r, acting from $L^2(Y_r)$ to $L^2(Y)$. Their result is if B_r is an operator of order $-1 + k$, satisfying the transmission condition, where $k = 0, 1, 2, \dots$, then there is a bound on the operator norm of B_r of the form Cr^{-k}, where C is independent of r. In particular, for $k = 0$, the B_r are uniformly bounded as $r \to 0$.

Unfortunately, this result does not quite give the result directly, since if we combine the operator bound Cr^{-2} for A_r and (6.4) for u and integrate in r, we encounter a logarithmic divergence. However, a small modification of this approach does the trick. If we let H^{-1} denote the inverse of the operator H on N, with Dirichlet boundary conditions, then H^{-1} is a pseudodifferential operator of order -2 when localized away from the boundary of N. Moreover, it satisfies the transmission condition. Since $Hu = \psi\delta_Y + \lambda u$, we may write

$$Au_{|Y} = AH^{-1}\big(\psi\delta_Y + \lambda u\big).$$

The first term is given by $(AH^{-1})_0\psi$ which is a L^2-bounded operator applied to ψ. By the upper bound for ψ, we see that this term satisfies (6.2). For the second term, we write u as the sum of a 'close' part and a 'far' part with respect to the boundary, relative to the length scale $\lambda^{-1/2}$. We can integrate in r as described above to get the bound for the close part, while a similar argument works for the far part. This completes the proof of the lower bound.

7. Periodic geodesics

In this section, we explore the relationship between trapped geodesics and the failure of the lower bound by considering two examples. The first is a hyperbolic cylinder, that is, the manifold $M = [-a, a] \times S^1_\theta$, with metric

$$g = dx^2 + (\cosh x)^2 d\theta^2.$$

This has a single periodic geodesic, at $x = 0$, which is unstable (geodesics are always unstable in manfolds with negative curvature). Let $\epsilon > 0$ be given, and let F be the set $\{|x| \geq \epsilon\}$; thus F excludes a neighbourhood of the trapped geodesic. Then Colin de Verdière and Parisse showed that there is a sequence of normalized Dirichlet eigenfunctions u_{k_j}, $k_j \to \infty$ as $j \to \infty$, such that

$$\int_F |u_{k_j}|^2 dg \sim \frac{1}{\log \lambda_{k_j}}, j \to \infty.$$

Then, applying our *upper* bound argument, which only uses the norm of eigenfunctions in a neighbourhood of the boundary (see the remark at the end of Section 4), we see that for some C

$$\|\psi_{k_j}\|_2^2 \leq C \frac{\lambda_{k_j}}{\log \lambda_{k_j}}.$$

This shows that the lower bound is violated. However, we can actually show that this is the true order of growth of $\|\psi_{k_j}\|_2^2$. Indeed, we shall show that for every normalized eigenfunction u,

$$\|\psi\|_2^2 \geq c \frac{\lambda}{\log \lambda}. \tag{7.1}$$

To show (7.1), we first observe that for every eigenfunction u, we have

$$\int_F |u|^2 dg \geq \frac{C}{\log \lambda}. \tag{7.2}$$

This follows by looking at a basis of eigenfunctions $u_{l,\lambda}$ of the form

$$u_{l,\lambda} = e^{il\theta} (\cosh x)^{-1/2} v_{l,\lambda}.$$

Then $v_{l,\lambda}$ satisfies

$$\Big(D_x^2 + \frac{1}{2} - \frac{1}{4} (\tanh x)^2 + l^2 (\operatorname{sech} x)^2 \Big) v_{l,\lambda} = \lambda v_{l,\lambda},$$

which we rewrite in the form

$$\big(h^2 D_x^2 + V \big) v_{l,\lambda} = E v_{l,\lambda}, \tag{7.3}$$

with

$$h^{-2} = l^2 + \frac{1}{4}, \quad V(x) = (\operatorname{sech} x)^2 - 1, \quad E = \frac{\lambda - l^2 - 1/2}{l^2 + 1/4}.$$

Then Theorem 20 of [2] shows that for $|E| < C$, (7.2) holds. For $E > C$, direct analysis of equation (7.3) shows that $v_{l,\lambda}$ has a uniform L^∞ bound, independent of h and E. Thus, in this case (7.2) holds *a fortiori*. In the remaining case, $E < -C < 0$, the origin is in the classically forbidden region and the result follows immediately from Agmon-type exponential decay estimates.

To complete the proof of (7.1), we construct an operator A which has a *nonnegative* commutator with H, and which is strictly positive in a neighbourhood of F. We are able to do this because of the special properties of geodesic flow on the hyperbolic cylinder. Letting G be the complement of F, and writing $F = F_+ \cup F_-$ for the two components of F, labelled according to the sign of x, a geodesic that passes from F_+, say, to G either stays over G for all subsequent time, or emerges into the region F_- and eventually reaches the boundary of M; it cannot happen that a geodesic starts in G, then moves into the set F, and returns to G. Thus, given a geodesic γ we can define an operator A with a symbol which is linearly increasing when γ is above the set F, and vanishing in some neighbourhood $U \subset\subset G$ of the periodic geodesic. As before, we can (by compactness) find a finite number of such operators whose sum has the desired property.

Thus, for a given eigenfunction u, let $M(u)$ denote the quantity on the left hand side of (7.2). If we go back to (6.1), then we find that the left hand side is at least as big as

$$c\lambda M(u) - C\lambda^{1/2} \geq c'\lambda M(u).$$

On the other hand, the argument above applied to Au shows that the right hand side is no bigger than

$$C\|\psi\|_2 \lambda^{1/2} M(u)^{1/2}.$$

We get the normalization factor $M(u)^{1/2}$ (instead of 1) since all arguments are localized near the boundary of M. The combination of these two estimates yields (7.1).

The second example we analyze is the spherical cylinder, that is, $M = (-a, a) \times S^1$, for $0 < a < 1$, with metric

$$g = dx^2 + (\cos x)^2 d\theta^2.$$

This has a periodic geodesic at $x = 0$. However, in this case, the geodesic is stable, and indeed every nearby geodesic is periodic. Thus, this case is the opposite extreme where there is an open set of periodic geodesics. We shall see that, correspondingly, the lower bound is violated in an extreme fashion.

This example may be analyzed in a similar way to the hyperbolic cylinder. We may separate variables, so there is a basis of eigenfunctions $u_{l,\lambda}$ of the form

$$e^{il\theta}(\cos x)^{-1/2} v_{l,\lambda}.$$

Then $v_{l,\lambda}$ satisfies the equation

$$\big(h^2 D_x^2 + V\big) v_{l,\lambda} = E v_{l,\lambda}, \tag{7.4}$$

where now

$$h^{-2} = l^2 - \frac{1}{4}, \quad V(x) = (\sec x)^2 - 1, \quad E = \frac{\lambda - l^2 + 1/2}{l^2 - 1/4}.$$

Here, V has a nondegenerate global mininum at $x = 0$. Let us consider a sequence of eigenfunctions u_{k_j} with $l_{k_j} \to \infty$ and $E(\lambda_{k_j}, l_{k_j}) \leq E_1$, where $0 < E_1 < V(a)$, so that the boundary, $|x| = a$, is in the classically forbidden region. Then Agmon-type exponential decay estimates (see [4], chapter 3) hold, giving for some $\epsilon > 0$

$$|u_{k_j}(x,\theta)| \leq C e^{-\epsilon \lambda_{k_j}^{1/2}}, \quad |x| \geq \delta > 0, \quad \text{for some } \epsilon > 0.$$

The upper bound argument then gives

$$\|\psi_{k_j}\|_2 \leq C e^{-\epsilon' \lambda_{k_j}^{1/2}}, \quad \text{for some } \epsilon' > 0,$$

so we actually have exponential *decrease*, rather than $O(\lambda^{1/2})$ increase, in the L^2 norm of the normal derivative for any such sequence of eigenfunctions.

8. Vector fields are not enough

Finally, we remark that the following example shows that one cannot expect to find a first order *differential* operator A having positive commutator with H.

First we analyze what it means for a vector field to have a positive commutator with H. Let the symbol of A be $a_i(x)\xi_i$. The Hamilton vector field of H is $\xi_i \partial_{x_i}$, so having a positive commutator requires that

$$\xi_i \xi_j \frac{\partial a_j}{\partial x_i} > 0 \text{ for } |\xi| \neq 0. \tag{8.1}$$

Thus, the matrix $\partial_{x_i} a_j$ must be positive definite, or in other words a_1 is increasing in direction e_1, etc.

Now consider the manifold with boundary shown in the figure (the corners should be assumed to have been smoothed out so that it has smooth boundary). In the figure, the topmost and bottom horizontal dotted lines, and the leftmost and rightmost dotted lines, are identified.

This manifold has no trapped geodesics.

Suppose that there is a vector field A whose commutator with H has a positive symbol. Notice that the two points p and q are such that there are three geodesics from p to q: one in the direction e_1+e_2, one in the direction $-e_1$ and one in the direction $-e_2$. Write $A = a_1e_1 + a_2e_2$. Then a_1 is increasing in direction e_1, and a_2 is increasing in direction e_2. Hence $a_1(p) > a_1(q)$, and $a_2(p) > a_2(q)$. On the other hand, $a_1 + a_2$ is increasing in direction e_1+e_2, so this yields $a_1(p)+a_2(p) < a_1(q)+a_2(q)$, which is a contradiction.

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A SYMMETRIC FUNCTIONAL CALCULUS FOR NONCOMMUTING SYSTEMS OF SECTORIAL OPERATORS

BRIAN JEFFERIES

Abstract. Given a system $A = (A_1, \dots, A_n)$ of linear operators whose real linear combinations have spectra contained in a fixed sector in $\mathbb{C}$ and satisfy resolvent bounds there, functions $f(A)$ of the system A of operators can be formed for monogenic functions f having decay at zero and infinity in a corresponding sector in $\mathbb{R}^{n+1}$. The paper discusses how the functional calculus $f \mapsto f(A)$ might be extended to a larger class of monogenic functions and its relationship with a functional calculus for analytic functions in a sector of $\mathbb{C}^n$.

1. Introduction

Given a finite system $A = (A_1, \dots, A_n)$ of bounded linear operators acting on a Banach space X, it has recently been shown how functions $f(A)$ of the n-tuple A can be formed for a large class of functions f, just under the assumption that the spectrum $\sigma(\langle A, \xi\rangle)$ of the operator $\langle A, \xi\rangle := \sum_{j=1}^{n} A_j\xi_j$ is a subset of $\mathbb{R}$ for every $\xi \in \mathbb{R}^n$ [5]. The operators $A_1, \dots, A_n$ do not necessarily commute with each other.

A distinguished subset $\gamma(A)$ of $\mathbb{R}^n$ with the property that the bounded linear operator $f(A)$ is defined for any real analytic function $f : U \to \mathbb{C}$ defined in a neighbourhood U of $\gamma(A)$ in $\mathbb{R}^n$ arises in the approach considered in [5]. For a polynomial p in n real variables, $p(A)$ is the operator formed by substituting symmetric products in the bounded linear operators $A_1, \dots, A_n$ for the monomial expressions in p, that is, we have a *symmetric* functional calculus in the n operators $A_1, \dots, A_n$. Another way of expressing this symmetry property is that for any $\xi \in \mathbb{R}^n$ and any polynomial $q : \mathbb{C} \to \mathbb{C}$ in one variable, the equality $p(A) = q(\langle A, \xi\rangle)$ holds for the polynomial $p : x \mapsto q(\langle x, \xi\rangle)$, $x \in \mathbb{R}^n$. Moreover, the mapping $f \mapsto f(A)$ is continuous for a certain topology

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defined on the space of functions real analytic in a neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$ [5, Proposition 3.3].

These properties are analogous to the Riesz-Dunford functional calculus of a single bounded linear operator T acting on X, by which a function $f(T) : X \to X$ of T is defined by the Cauchy integral formula

$$f(T) = \frac{1}{2\pi i} \int_C (\lambda I - T)^{-1} f(\lambda)\, d\lambda, \tag{1}$$

for f analytic in a neighbourhood of the spectrum $\sigma(T)$ of T and for C a simple closed contour about $\sigma(T)$. It is by this analogy that the set $\gamma(A)$ mentioned above may be thought of as the "joint spectrum" of the system A, especially if there is no set smaller than $\gamma(A)$ possessing the desirable properties alluded to.

Now for a single operator T, the spectrum $\sigma(T)$ has a simple algebraic definition as the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is not invertible in the space $\mathcal{L}(X)$ of bounded linear operators acting on X. In the case that A consists of a system of n *commuting*, possibly unbounded, linear operators with real spectra, A. McIntosh and A. Pryde [14, 15] gave a simple algebraic definition of the joint spectrum $\gamma(A)$ of A and used this to obtain operator bounds for solutions of operator equations. Work of A. McIntosh, A. Pryde and W. Ricker [16] established the equivalence of $\gamma(A)$ with other notions of *joint spectrum*.

In the noncommutative case, we cannot expect such a straightforward algebraic definition of the joint spectrum $\gamma(A)$, although such a definition was proposed in [8]. Another example of a symmetric functional calculus is the *Weyl calculus* $\mathcal{W}_A$ considered in [19] for n selfadjoint operators $A = (A_1, \dots, A_n)$. In the case that the system A consists of bounded selfadjoint operators, it was shown in [4] that $\gamma(A)$ is precisely the support of the operator valued distribution $\mathcal{W}_A$, and E. Nelson characterised this set as the Gelfand spectrum of a certain subalgebra of the Banach algebra of *operants* [17]. Further work along these lines was conducted by E. Albrecht [1].

If we now pass to unbounded operators, then a similar analysis holds if we retain the spectral reality condition $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for $\xi \in \mathbb{R}^n$, provided that we suitably account for operator domains. However, much of the work [14, 15, 16] on functional calculi just mentioned was motivated by Alan McIntosh's study of the commuting n-tuple $D_\Sigma = (D_1, \dots, D_n)$ of differentiation operators on a Lipschitz surface Σ in $\mathbb{R}^{n+1}$. In the case that Σ is just the flat surface $\mathbb{R}^n$, the operators $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$, commute with each other and are selfadjoint, otherwise, the unbounded operators D_j, $j = 1, \dots, n$, have spectra $\sigma(D_j)$ contained in a complex sector $S_\omega(\mathbb{C}) = \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leq \omega\}$ with an angle ω depending on the variation of the directions normal to the surface Σ.

Although the existence and properties of the H^∞-functional calculus for the commuting n-tuple D_Σ are now well-understood, see for example [13], the purpose of the present paper is to initiate a study of the symmetric functional calculus for an n-tuple A of unbounded sectorial operators — we do not assume that the operators commute with each other. In particular, the spectral reality condition

$$\sigma(\langle A, \xi\rangle) \subset \mathbb{R}, \quad \text{for all } \xi \in \mathbb{R}^n \tag{2}$$

needs to be relaxed. An alternative approach to forming an H^∞-functional calculus for commuting operators using exponential bounds is given in [11].

Before proceeding with further discussion, we note the definition of the joint spectrum $\gamma(A)$ for a system A satisfying condition (2). The key idea behind [14, 15] in the commuting case and [4, 5, 8, 9] in the noncommuting case, is to produce a higher-dimensional analogue of the Riesz-Dunford formula (1). So what we need is a higher-dimensional analogue of the Cauchy integral formula in complex analysis and then, in the time-honoured fashion of operator theory, substitute an n-tuple of numbers by an n-tuple of unbounded linear operators. But this is easier said than done.

It turns out that Clifford analysis provides a higher dimensional analogue of the Cauchy integral formula especially well-suited to the noncommutative setting. Even for the commuting n-tuple D_Σ of operators mentioned above, it provides the connection between multiplier operators and singular convolution operators for functions defined on a Lipschitz surface. A brief résumé of Clifford analysis [2, 3] and the monogenic functional calculus treated in [5] follows.

Let $\mathbb{C}_{(n)}$ be the *Clifford algebra* generated over the field $\mathbb{C}$ by the standard basis vectors $e_0, e_1, \dots, e_n$ of $\mathbb{R}^{n+1}$ with conjugation $u \mapsto \overline{u}$. The *generalized Cauchy-Riemann operator* is given by $D = \sum_{j=0}^n e_j \frac{\partial}{\partial x_j}$.

Let $U \subset \mathbb{R}^{n+1}$ be an open set. A function $f : U \to \mathbb{C}_{(n)}$ is called *left monogenic* if $Df = 0$ in U and *right monogenic* if $fD = 0$ in U. The *Cauchy kernel* is given by

$$G_x(y) = \frac{1}{\sigma_n} \frac{\overline{x - y}}{|x - y|^{n+1}}, \quad x, y \in \mathbb{R}^{n+1}, x \neq y, \tag{3}$$

with $\sigma_n = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ the volume of unit n-sphere in $\mathbb{R}^{n+1}$. So, given a left monogenic function $f : U \to \mathbb{C}_{(n)}$ defined in an open subset U of $\mathbb{R}^{n+1}$ and an open subset Ω of U such that the closure $\overline{\Omega}$ of Ω is contained in U, and the boundary $\partial\Omega$ of Ω is a smooth oriented n-manifold, then the Cauchy integral formula

$$f(y) = \int_{\partial\Omega} G_x(y)\boldsymbol{n}(x)f(x)\,d\mu(x), \quad y \in \Omega$$

is valid. Here $\boldsymbol{n}(x)$ is the outward unit normal at $x \in \partial\Omega$ and μ is the volume measure of the oriented manifold $\partial\Omega$. An element $x = (x_0, x_1, \ldots, x_n)$ of $\mathbb{R}^{n+1}$ will often be written as $x = x_0e_0 + \vec{x}$ with $\vec{x} = \sum_{j=1}^n x_je_j$.

By analogy with formula (1), our aim is to define

$$f(A) = \int_{\partial\Omega} G_x(A)\boldsymbol{n}(x)f(x)\,d\mu(x) \tag{4}$$

for the n-tuple $A = (A_1, \ldots, A_n)$ of bounded linear operators on X. A difficulty occurs in making sense of the Cauchy kernel $x \mapsto G_x(A)$, a function with values in the space $\mathcal{L}(X) \otimes \mathbb{C}_{(n)}$ that should be defined and two-sided monogenic for all x off a nonempty closed subset $\gamma(A)$ of $\mathbb{R}^n$ inside Ω. The set $\partial\Omega$ can be smoothly varied in the region where $x \mapsto G_x(A)$ is right-monogenic. Of course, one would also like $f(A)$ to be the 'correct' operator in the case that f is the unique monogenic extension to $\mathbb{R}^{n+1}$ of a polynomial in n variables.

In the Riesz-Dunford functional calculus for T, the set of singularities of the resolvent $\lambda \mapsto (\lambda I - T)^{-1}$ is precisely the spectrum $\sigma(T)$ of T, so the set $\gamma(A)$ may be interpreted as a higher-dimensional analogue of the spectrum of a single operator. It seems reasonable to call the set $\gamma(A)$ the *monogenic spectrum* of the n-tuple A by analogy with the case of a single operator.

The program was implemented by A. McIntosh and A. Pryde for commuting n-tuples of bounded operators with real spectrum in order to give estimates on the solution of systems of operator equations [14, 15]. In the case that n is odd, we have

$$\gamma(A) = \left\{ \lambda \in \mathbb{R}^n : \sum_{j=1}^n (\lambda_j I - A_j)^2 \text{ is invertible in } \mathcal{L}(X) \right\}^c$$

and

$$G_x(A) = \frac{1}{\sigma_n}(\overline{x - A}) \left(x_0^2 I + \sum_{j=1}^n (x_j I - A_j)^2 \right)^{-\frac{n+1}{2}},$$

for all $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus (\{0\} \times \gamma(A))$. It turns out that $\gamma(A)$ coincides with the Taylor spectrum for commuting systems of bounded linear operators [16].

If the n-tuple A of bounded linear operators satisfies exponential growth conditions, such as when $A_1, \dots, A_n$ are selfadjoint, then Weyl's functional calculus $\mathcal{W}_A$ is associated with A and $G_x(A) = \mathcal{W}(G_x)$ is an obvious way to define the Cauchy kernel for all x outside the support of $\mathcal{W}_A$. It is shown in [4] that formula (4) holds. However, in this case, we actually have a symmetric functional calculus defined over $\gamma(A)$ richer than just all real analytic functions.

The Cauchy kernel $G_x(A)$ can also be written as a series expansion like the Neuman series for the resolvent of a single operator if $x \in \mathbb{R}^{n+1}$ lies outside a sufficiently large ball [8, 9], but the expansion does not allow us to identify $\gamma(A)$ as a subset of $\mathbb{R}^n$ in the case that the spectral reality condition (2) holds.

A third way to define the Cauchy kernel $G_x(A)$ for the monogenic functional calculus whenever the spectral reality condition (2) holds, is by the plane wave decomposition for the Cauchy kernel (3) given by F. Sommen [18]. This was investigated by A. McIntosh and J. Picton-Warlow soon after the papers [14, 15] appeared. The formula is

$$\begin{aligned} G_x(A) = {} & \frac{(n-1)!}{2} \left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(x_0)^{n-1} \\ & \times \int_{S^{n-1}} (e_0 + is) \left(\langle \vec{x}, s\rangle I - \langle A, s\rangle - x_0 s I\right)^{-n} ds \end{aligned} \tag{5}$$

for all $x = x_0 e_0 + \vec{x}$ with x_0 a nonzero real number and $\vec{x} \in \mathbb{R}^n$. Here S^{n-1} is the unit $(n-1)$-sphere in $\mathbb{R}^n$, ds is surface measure and the inverse power $(\langle \vec{x} I - A, s\rangle - x_0 s)^{-n}$ is taken in the Clifford module $\mathcal{L}(X) \otimes C_{(n)}$. The spectral reality condition (2) ensures the invertibility of $(\langle \vec{x} I - A, s\rangle - x_0 s)$ for all $x_0 \neq 0$ and $s \in S^{n-1}$ by the spectral mapping theorem.

Even if A satisfies exponential growth conditions, with the left hand side given by formula (5), the equality $G_x(A) = \mathcal{W}_A(G_x)$ can still be used to good effect. In [6], it was used to geometrically characterise the support of fundamental solution of the symmetric hyperbolic system associated with a pair A of hermitian matrices in the case $n = 2$. Greater understanding of Clifford residue theory would enable a similar treatment in higher dimensions.

In Section 2, it is shown how formula (5) for the Cauchy kernel associated with the system A of sectorial operators still works if the spectral reality condition (2) is replaced by a sectoriality condition with the appropriate resolvent bounds. The system D_Σ of commuting sectorial operators described above is of this type. By this means functions $f(A)$ of the operators A can be formed, provided that f is, say, left monogenic in a sector in $\mathbb{R}^{n+1}$ and satisfies suitable decay estimates at 0 and ∞, in a fashion similar to the case of a single operator of type ω [12]. Because $G_x(A)$ is only defined for x outside a sector in $\mathbb{R}^{n+1}$, the monogenic spectrum $\gamma(A)$ is now contained in that sector in $\mathbb{R}^{n+1}$. Recall that under condition (2), $\gamma(A)$ is a subset of $\mathbb{R}^n$.

A function $f(D_\Sigma)$ of the system D_Σ has a natural interpretation as a multiplier operator acting on L^p-spaces of functions defined on the Lipschitz surface Σ, as well as a singular convolution operator, so the multiplier f should be a bounded analytic function defined on a sector in $\mathbb{C}^n$ [13], rather than, say, a bounded monogenic function defined in a sector in $\mathbb{R}^{n+1}$. The monogenic functional calculus for a system A of sectorial operators appears to be moving us inexorably in the wrong direction.

The problem arises of establishing a bijection between monogenic functions defined on a sector in $\mathbb{R}^{n+1}$ and analytic functions defined on a sector in $\mathbb{C}^n$, together with the appropriate norms — this is a question of function theory rather than operator theory. The association is via the Cauchy-Kowaleski extension to a sector in $\mathbb{R}^{n+1}$ of the restriction of the analytic function to $\mathbb{R}^n \setminus \{0\}$.

The purpose of this paper is to make some observations about the relationship between monogenic functions defined in a sector in $\mathbb{R}^{n+1}$ and analytic functions defined in the corresponding sector in $\mathbb{C}^n$, with applications to functional calculi of systems of operators firmly in mind. In Section 3, the spectral properties of multiplication operators in $\mathbb{C}_{(n)}$ are examined along the lines of Lecture 1 of [13]. In Section 4, this enables us to uniquely associate a bounded analytic function defined in a sector in $\mathbb{C}^n$ with a suitably decaying monogenic function defined in the corresponding sector in $\mathbb{R}^{n+1}$ via the Cauchy integral formula.

2. The plane wave decomposition

Let $A = (A_1, \dots, A_n)$ be an n-tuple of densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in X such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in X. The space $\mathcal{L}_{(n)}(X_{(n)})$ of left module homomorphisms of $X_{(n)} = X \otimes \mathbb{C}_{(n)}$ is identified with $\mathcal{L}(X) \otimes \mathbb{C}_{(n)}$ in the natural way and becomes a right Banach module under the uniform operator topology.

If we take formula (5) as the definition of $G_x(A)$, then the convergence of the integral

$$\int_{S^{n-1}} (e_0 + is) \left(\langle \vec{x}I - A, s\rangle - x_0 sI\right)^{-n} ds$$

for particular values of $x = x_0e_0 + \vec{x} \in \mathbb{R}^{n+1}$ is at issue. Now

$$\left(\langle \vec{x}I - A, s\rangle - x_0 sI\right)^{-1} = \left(\langle \vec{x}I - A, s\rangle + x_0 sI\right)\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2 I\right)^{-1}$$

if $0 \notin \sigma\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2\right)$. Thus, we need to ensure the appropriate uniform operator bounds for

$$\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2 I\right)^{-1}, \quad s \in S^{n-1}$$

as $x = x_0e_0 + x$ ranges over a subset of $\mathbb{R}^{n+1}$. In the case that $\sigma(\langle A, s\rangle) \subset \mathbb{R}$ and $(\lambda I - \langle A, s\rangle)^{-1}$ is suitably bounded for all $s \in S^{n-1}$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then $G_{x_0e_0+\vec{x}}(A)$ is defined for all $x_0 \neq 0$. First, for each $0 < \nu < \pi/2$, set

$$\begin{aligned}
S_{\nu+}(\mathbb{C}) &= \{z \in \mathbb{C} : |\arg z| \le \nu\} \cup \{0\},\\
S_{\nu}(\mathbb{C}) &= S_{\nu+}(\mathbb{C}) \cup \underline{i}g\big(-S_{\nu+}(\mathbb{C})\big),\\
S^{\circ}_{\nu+}(\mathbb{C}) &= \{z \in \mathbb{C} : |\arg z| < \nu\},\\
S^{\circ}_{\nu}(\mathbb{C}) &= S^{\circ}_{\nu+}(\mathbb{C}) \cup \big(-S^{\circ}_{\nu+}(\mathbb{C})\big).
\end{aligned}$$

The $(n-1)$-sphere in $\mathbb{R}^n$ is denoted by S^{n-1}.

Definition 2.1. Let $A = (A_1, \dots, A_n)$ be an n-tuple of densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in X such that $\bigcap_{j=1}^n \mathcal{D}(A_j)$ is dense in X and let $0 \le \omega < \frac{\pi}{2}$. Then A is said to be *uniformly of type* ω if for every $s \in S^{n-1}$, the operator $\langle A, s\rangle$ is closable with closure $\overline{\langle A, s\rangle}$, the inclusion $\sigma\big(\overline{\langle A, s\rangle}\big) \subset S_\omega(\mathbb{C})$ holds, and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\|\big(zI - \overline{\langle A, s\rangle}\big)^{-1}\| \le C_\nu |z|^{-1}, \quad z \notin S^{\circ}_{\nu}(\mathbb{C}),\ s \in S^{n-1}. \tag{6}$$

It follows that $s \mapsto \overline{\langle A, s\rangle}$ is continuous on S^{n-1} in the sense of strong resolvent convergence [7, Theorem VIII.1.5]. Because $\big(zI - \langle A, s\rangle\big)^{-1}$ is densely defined and uniformly bounded in X, the closure symbol will be omitted.

Now suppose that equation (6) is satisfied and let $z = \langle \vec{x}, s\rangle + ix_0$. Then $z \notin S^{\circ}_{\nu}(\mathbb{C})$ means that $|\arg z| \ge \nu$ for $-\frac{\pi}{2} \le \arg z \le \frac{\pi}{2}$ or $\pi - \arg z \ge \nu$ for $\frac{\pi}{2} \le \arg z \le \pi$ or $\pi + \arg z \ge \nu$ for $-\pi \le \arg z \le -\frac{\pi}{2}$. Hence, we have $|x_0| \ge \tan\nu |\langle \vec{x}, s\rangle|$.

First, let

$$\begin{aligned} N_\nu &= \{x \in \mathbb{R}^{n+1} : x = x_0 e_0 + \vec{x},\ |x_0| \geq \tan \nu |\vec{x}|\ \}, \\ S_\nu(\mathbb{R}^{n+1}) &= \{x \in \mathbb{R}^{n+1} : x = x_0 e_0 + \vec{x},\ |x_0| \leq \tan \nu |\vec{x}|\ \}, \\ S_\nu^\circ(\mathbb{R}^{n+1}) &= \{x \in \mathbb{R}^{n+1} : x = x_0 e_0 + \vec{x},\ |x_0| < \tan \nu |\vec{x}|\ \}. \end{aligned}$$

Note that if $x_0 e_0 + x \in N_\nu$, then $z = \langle \vec{x}, s\rangle + i x_0 \notin S_\nu^\circ(\mathbb{C})$ for every $s \in S^{n-1}$, because either $|x_0| \geq \tan \nu |\vec{x}| \geq \tan \nu |\langle \vec{x}, s\rangle|$.

Lemma 2.2. *Let $\omega < \nu < \pi/2$. Suppose that the n-tuple A of linear operators is uniformly of type ω. Then for all $x_0 e_0 + \vec{x} \in N_\nu$, the integral*

$$\int_{S^{n-1}} \left\| (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-n} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} ds$$

converges and satisfies the bound

$$\int_{S^{n-1}} \left\| (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-n} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} ds \leq \frac{C'_\nu}{|x_0|^n}.$$

Proof. For every $x_0 e_0 + \vec{x} \in N_\nu$, we have $z = \langle x, s\rangle \pm i x_0 \notin S_\nu(\mathbb{C})$ so that the operator $(\langle \vec{x}, s\rangle \pm i x_0) I - \langle A, s\rangle$ is invertible and the bound

$$\left\| ((\langle \vec{x}, s\rangle \pm i x_0) I - \langle A, s\rangle)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{C_\nu}{\sqrt{\langle \vec{x}, s\rangle^2 + x_0^2}}$$

holds. Now

$$\begin{aligned} &(\langle \vec{x} I - A, s\rangle - x_0 s I)^{-1} \\ &\qquad = (\langle \vec{x} I - A, s\rangle + x_0 s I) \left(\langle \vec{x} I - A, s\rangle^2 + x_0^2 I\right)^{-1} \end{aligned}$$

where

$$\begin{aligned} &\left(\langle \vec{x} I - A, s\rangle^2 + x_0^2 I\right)^{-1} \\ &\qquad = ((\langle \vec{x}, s\rangle + i x_0) I - \langle A, s\rangle)^{-1} ((\langle \vec{x}, s\rangle - i x_0) I - \langle A, s\rangle)^{-1}. \end{aligned}$$

Writing $(\langle \vec{x} I - A, s\rangle + x_0 s I) = ((\langle \vec{x}, s\rangle + i x_0) I - \langle A, s\rangle) - i x_0 I + x_0 s I$, we obtain

$$\begin{aligned} (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-1} &= ((\langle \vec{x}, s\rangle - i x_0) I - \langle A, s\rangle)^{-1} \\ &\qquad - i x_0 (e_0 + i s) \left(\langle \vec{x} I - A, s\rangle^2 + x_0^2 I\right)^{-1}, \end{aligned}$$

so that by the estimate (6) we have

$$\begin{aligned} \left\| (\langle \vec{x} I - A, s\rangle - x_0 s I)^{-1} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} &\leq \frac{C_\nu}{\sqrt{\langle \vec{x}, s\rangle^2 + x_0^2}} + \frac{2|x_0| C_\nu^2}{\langle \vec{x}, s\rangle^2 + x_0^2} \\ &\leq \frac{C_\nu + 2C_\nu^2}{|x_0|}, \end{aligned}$$

from which the stated bound follows. □

Thus, if A is uniformly of type ω, then $x_0e_0 + \vec{x} \mapsto G_{x_0e_0+\vec{x}}(A)$ is defined by equation (5) for all $x_0e_0 + \vec{x} \in N_\nu$ with $\omega < \nu < \pi/2$. Standard arguments ensure that $x_0e_0 + \vec{x} \mapsto G_{x_0e_0+\vec{x}}(A)$ is both left and right monogenic as an element of $\mathcal{L}(X) \otimes \mathbb{C}_{(n)}$. If we denote by $\gamma(A) \subset \mathbb{R}^{n+1}$ the set of all singularities of the function $x_0e_0 + \vec{x} \mapsto G_{x_0e_0+\vec{x}}(A)$, then

$$\gamma(A) \subseteq S_\omega(\mathbb{R}^{n+1}).$$

Suppose that $\omega < \nu < \pi/2$, $0 < s < n$ and f is a left monogenic function defined on $S_\nu^\circ(\mathbb{R}^{n+1})$ such that for every $0 < \theta < \nu$ there exists $C_\theta > 0$ such that

$$|f(x)| \le C_\theta \frac{|x|^s}{(1+|x|^{2s})}, \quad x \in S_\theta^\circ(\mathbb{R}^{n+1}). \tag{7}$$

According to Lemma 2.2, for every $\omega < \nu' < \theta < \nu$, we have

$$\|G_x(A)\|.|f(x)| \le C_{\theta,\nu'} \frac{|x|^s}{|x_0|^n(1+|x|^{2s})}, \quad x = x_0e_0 + \vec{x}$$

for all $x \in S_\theta^\circ(\mathbb{R}^{n+1}) \cap N_{\nu'}$.

Now if $\omega < \theta < \nu$ and

$$H_\theta = \{x \in \mathbb{R}^{n+1} : x = x_0e_0 + \vec{x},\ |x_0|/|x| = \tan\theta\} \subset S_\nu^\circ(\mathbb{R}^{n+1}). \tag{8}$$

it follows that $\|G_x(A)\|.|f(x)| = O(1/|x|^{n-s})$ as $x \to 0$ in H_θ. Hence, $x \mapsto G_x(A)\boldsymbol{n}(x)f(x)$ is locally integrable at zero with respect to n-dimensional surface measure on H_θ. Similarly, $\|G_x(A)\|.|f(x)| = O(1/|x|^{n+s})$ as $|x| \to \infty$ in H_θ, so $x \mapsto G_x(A)\boldsymbol{n}(x)f(x)$ is integrable with respect to n-dimensional surface measure on H_θ.

Therefore, we define

$$f(A) = \int_{H_\theta} G_x(A)\boldsymbol{n}(x)f(x)\,d\mu(x). \tag{9}$$

If $\psi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ has a two-sided monogenic extension $\tilde{\psi}$ to $S_\nu^\circ(\mathbb{R}^{n+1})$ that satisfies the bound (7) for all $0 < \theta < \nu$, then $\tilde{\psi}(A)$ is written just as $\psi(A)$.

Formula (9) does just what we would expect in the noncommuting situation. For example, let p be a polynomial of degree n with $p(0) = 0$ and $b_\lambda(z) = p(z)(\lambda - z)^{-n-1}$ for some $\lambda \notin S_\nu^\circ(\mathbb{C})$. Let $\xi \in \mathbb{R}^n$ and set $\phi_{\lambda,\xi}(x) = b_\lambda(\langle x, \xi\rangle)$ for all $x \in \mathbb{R}^n$. Denote the two-sided monogenic extension of $\phi_{\lambda,\xi}$ to $S_\nu^\circ(\mathbb{R}^{n+1})$ by $\tilde{\phi}_{\lambda,\xi}$. Then $\tilde{\phi}_{\lambda,\xi}$ has decay at zero and infinity and we have $\phi_{\lambda,\xi}(A) = \tilde{\phi}_{\lambda,\xi}(A) = p(\langle A, \xi\rangle)(\lambda I - \langle A, \xi\rangle)^{-n-1}$ is a bounded linear operator.

In order to form functions $f(A)$ of the system A of operators for a class of monogenic functions f larger than those which satisfy a bound like (7), a greater understanding of function theory in the sector $S_\omega(\mathbb{R}^{n+1})$ is needed. To this end, the simple system $A = \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ of multiplication operators in the algebra $\mathbb{C}_{(n)}$ is considered in the next section.

3. Joint spectral theory in the algebra $\mathbb{C}_{(n)}$

Let $\zeta = (\zeta_1, \dots, \zeta_n)$ be a vector belonging to $\mathbb{C}^n$. The complex spectrum $\sigma(i\zeta)$ of the element $i\zeta = i(\zeta_1 e_1 + \cdots + \zeta_n e_n)$ of the algebra $\mathbb{C}_{(n)}$ is

$$\sigma(i\zeta) = \{\lambda \in \mathbb{C} : (\lambda e_0 - i\zeta) \text{ does not have an inverse in } \mathbb{C}_{(n)}\}.$$

Following [13, Section 5.2], we check that

$$(\lambda e_0 + i\zeta)(\lambda e_0 - i\zeta) = \lambda^2 e_0 - i^2\zeta^2 = (\lambda^2 - |\zeta|^2_{\mathbb{C}})e_0,$$

where $|\zeta|^2_{\mathbb{C}} = \sum_{j=1}^n \zeta_j^2$. So, for all $\lambda \in \mathbb{C}$ for which, $\lambda \neq \pm|\zeta|_{\mathbb{C}}$, the element $(\lambda e_0 - i\zeta)$ of the algebra $\mathbb{C}_{(n)}$ is invertible and

$$(\lambda e_0 - i\zeta)^{-1} = \frac{\lambda e_0 + i\zeta}{\lambda^2 - |\zeta|^2_{\mathbb{C}}}.$$

If $|\zeta|^2_{\mathbb{C}} \neq 0$, the two square roots of $|\zeta|^2_{\mathbb{C}}$ are written as $\pm|\zeta|_{\mathbb{C}}$ and $|\zeta|_{\mathbb{C}} = 0$ for $|\zeta|^2_{\mathbb{C}} = 0$. Hence $\sigma(i\zeta) = \{\pm|\zeta|_{\mathbb{C}}\}$. When $|\zeta|^2_{\mathbb{C}} \neq 0$, the spectral projections

$$\chi_\pm(\zeta) = \frac{1}{2}\left(e_0 + \frac{i\zeta}{\pm|\zeta|_{\mathbb{C}}}\right)$$

are associated with each element $\pm|\zeta|_{\mathbb{C}}$ of the spectrum $\sigma(i\zeta)$ and $i\zeta$ has the spectral representation $i\zeta = |\zeta|_{\mathbb{C}}\chi_+(\zeta) + (-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$. Henceforth, we use the symbol $|\zeta|_{\mathbb{C}}$ to denote the positive square root of $|\zeta|^2_{\mathbb{C}}$ in the case that $|\zeta|^2_{\mathbb{C}} \notin (- \in fty, 0]$.

On the other hand, according to the point of view mentioned in the Introduction, the *monogenic spectrum* $\gamma(\zeta)$ of $\zeta \in \mathbb{C}^n$ should be the set of singularities of the Cauchy kernel $x \mapsto G_x(\zeta)$ in the algebra $\mathbb{C}_{(n)}$. Although $G_x(\zeta)$ is defined by formula (3) only for $\zeta \in \mathbb{R}^n$ and $x \neq \zeta$, a natural choice for the Cauchy kernel for $\zeta \in \mathbb{C}^n$ is to take the maximal analytic extension $\zeta \mapsto G_x(\zeta)$ of formula (3) for $\zeta \in \mathbb{C}^n$, that is,

(10)
$$G_x(\zeta) = \frac{1}{\sigma_n}\frac{\overline{x} + \zeta}{|x - \zeta|^{n+1}_{\mathbb{C}}}, \quad x \in \mathbb{R}^{n+1}, \quad \begin{cases} |x-\zeta|^2_{\mathbb{C}} \notin (-\infty, 0], & n \text{ even} \\ |x-\zeta|^2_{\mathbb{C}} \neq 0, & n \text{ odd} \end{cases}$$

Here $|x - \zeta|^2_{\mathbb{C}} = x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2$ and $|x - \zeta|_{\mathbb{C}}$ is the positive square root of $|x - \zeta|^2_{\mathbb{C}}$, coinciding with the analytic extension of the modulus function $\xi \mapsto |x - \xi|$, $\xi \in \mathbb{R}^n \setminus \{x\}$.

The analogous reasoning for multiplication by $x \in \mathbb{R}^{n+1}$ in the algebra $\mathbb{C}_{(n)}$ just gives us the Cauchy kernel (3), so that $\gamma(x) = \{x\}$, as expected.

Remark 3.1. If $\zeta = (\zeta_1, \dots, \zeta_n)$ satisfies the conditions of Definition 2.1, then there exists $\theta \in [-\omega, \omega]$ and $x \in \mathbb{R}^n$ such that $\zeta = e^{i\theta}x$. To see this, write $\zeta = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}^n$. If $|\langle \beta, \xi\rangle| \le |\langle \alpha, \xi\rangle| \tan\omega$ for all $\xi \in \mathbb{R}^n$, then $\alpha^\perp \subset \beta^\perp$, so that $\beta \in \operatorname{span}\{\alpha\}$.

In this case, the plane wave formula (5) with $A = \zeta$ and equation (10) agree by analytic continuation, at least for $x \in N_\nu$ with $\nu > |\theta|$.

Given $\zeta \in \mathbb{C}^n$, if singularities of (10) occur at $x \in \mathbb{R}^{n+1}$, then $|x - \zeta|^2_{\mathbb{C}} \in (-\infty, 0]$, otherwise we can simply take the positive square root of $|x - \zeta|^2_{\mathbb{C}}$ in formula (10) to obtain a monogenic function of x. To determine this set, write $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbb{R}^n$ and $x = x_0e_0 + \vec{x}$ for $x_0 \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$. Then

$$\begin{aligned} |x-\zeta|^2_{\mathbb{C}} &= x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2 \\ &= x_0^2 + \sum_{j=1}^n (x_j - \xi_j - i\eta_j)^2 \\ &= x_0^2 + |\vec{x} - \xi|^2 - |\eta|^2 - 2i\langle \vec{x} - \xi, \eta\rangle. \end{aligned} \tag{11}$$

Thus, $|x-\zeta|^2_{\mathbb{C}}$ belongs to $(-\infty, 0]$ if and only if x lies in the intersection of the hyperplane $\langle \vec{x} - \xi, \eta\rangle = 0$ passing through ξ and with normal η, and the ball $x_0^2 + |\vec{x} - \xi|^2 \le |\eta|^2$ centred at ξ with radius $|\eta|$. If n is even, then

$$\gamma(\zeta) = \{x = x_0e_0 + \vec{x} \in \mathbb{R}^{n+1} : \langle \vec{x} - \xi, \eta\rangle = 0,\ x_0^2 + |\vec{x} - \xi|^2 \le |\eta|^2\}. \tag{12}$$

and if n is odd, then

$$\gamma(\zeta) = \{x = x_0e_0 + \vec{x} \in \mathbb{R}^{n+1} : \langle \vec{x} - \xi, \eta\rangle = 0,\ x_0^2 + |\vec{x} - \xi|^2 = |\eta|^2\}. \tag{13}$$

In particular, if $\Im(\zeta) = 0$, then $\gamma(\zeta) = \{\zeta\} \subset \mathbb{R}^n$.

Remark 3.2. The distinction between n odd and even is reminiscent of the support of the fundamental solution of the wave equation in even and odd dimensions.

Off $\gamma(\zeta)$, the function $x \mapsto G_x(\zeta)$ is clearly two-sided monogenic, so the Cauchy integral formula gives

$$\tilde{f}(\zeta) = \int_{\partial\Omega} G_x(\zeta)\boldsymbol{n}(x)f(x)\,d\mu(x) \tag{14}$$

for a bounded open neighbourhood Ω of $\gamma(\zeta)$ with smooth oriented boundary $\partial\Omega$, outward unit normal $\boldsymbol{n}(x)$ at $x \in \partial\Omega$ and surface measure μ. The function f is assumed to be left monogenic in a neighbourhood of $\overline{\Omega}$, but $\zeta \mapsto \tilde{f}(\zeta)$ is an analytic $\mathbb{C}_{(n)}$-valued function as the closed set $\gamma(\zeta)$ varies inside Ω. Moreover, $\tilde{f}$ equals f on $\Omega \cap \mathbb{R}^n$ by the usual Cauchy integral formula of Clifford analysis, so if f is, say, the monogenic extension of a polynomial $p : \mathbb{C}^n \to \mathbb{C}$ restricted to $\mathbb{R}^n$, then $\tilde{f}(\zeta) = p(\zeta)$, as expected. In this way, for each left monogenic function f defined in a neighbourhood of $\gamma(\zeta)$, in a natural way we associate an analytic function $\tilde{f}$ defined in a neighbourhood of ζ.

It is clear that if $\zeta = \xi + i\eta$ lies in a sector in $\mathbb{C}^n$, say, $|\eta| \le |\xi| \tan\nu$, then the monogenic spectrum $\gamma(\zeta)$ lies in a corresponding sector in $\mathbb{R}^{n+1}$. More precisely, we have

Proposition 3.3. *Let* $\zeta \in \mathbb{C}^n \setminus \{0\}$ *and* $0 < \omega < \pi/2$. *Then* $\gamma(\zeta) \subset S_\omega(\mathbb{R}^{n+1})$ *if and only if*

$$|\zeta|^2_{\mathbb{C}} \ne (-\infty, 0] \quad \text{and} \quad |\Im(\zeta)| \le \Re(|\zeta|_{\mathbb{C}}) \tan\omega. \tag{15}$$

Proof. The statement is trivially valid if $\zeta \in \mathbb{R}^n \setminus \{0\}$, so suppose that $\Im(\zeta) \ne 0$. Then the monogenic spectrum $\gamma(\zeta)$ of ζ given by (12) is a subset of $S_\omega(\mathbb{R}^{n+1})$ if and only if there exists $0 < \theta \le \omega$ such that the cone

$$H^+_\theta = \{x_0e_0 + \vec{x} \in \mathbb{R}^{n+1} : x_0 > 0,\ x_0 = |x| \tan\theta\ \}$$

is tangential to the boundary of $\gamma(\zeta)$. A calculation shows that H^+_θ is tangential to the boundary of $\gamma(\zeta)$ for all $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$, satisfying

$$|\eta|^2 = \sin^2\theta(|\xi|^2 + \tan^2\theta |P_\eta \xi|^2). \tag{16}$$

Here $P_\eta : u \mapsto \langle u, \eta\rangle \eta/|\eta|^2$, $u \in \mathbb{R}^n$, is the projection operator onto $\operatorname{span}\{\eta\}$.

To relate condition (16) to the inequality (15), suppose that $m = m_0e_0 + \vec{m}$ is the unit vector normal to H_θ such that $\vec{m}$ lies in the direction of η. Hence, $m_0 = \cot\theta |\vec{m}|$, $\tan\theta = |\vec{m}|/m_0$ and $P_\eta\xi = \langle \xi, \vec{m}\rangle \vec{m}/|\vec{m}|^2$. Then equation (15) becomes

$$\eta = \sin\theta(m_0^2|\xi|^2 + \langle \xi, \vec{m}\rangle^2)^{1/2} \frac{\vec{m}}{|\vec{m}| m_0}.$$

But $|m_0e_0 + \vec{m}| = 1$, so $(\cot^2\theta + 1)|\vec{m}|^2 = 1$. We have $|\vec{m}| = \sin\theta$ and

$$\eta = (m_0^2|\xi|^2 + \langle \xi, \vec{m}\rangle^2)^{1/2} \frac{\vec{m}}{m_0}. \tag{17}$$

As mentioned in [13, p67], the set of all $\zeta = \xi + i\eta$ with $\eta \neq 0$ satisfying (17) is equal to the set of all $\zeta = \xi + i\eta$ with $\eta \neq 0$ satisfying

$$|\zeta|^2_{\mathbb{C}} \neq (-\infty, 0] \quad \text{and} \quad \eta = \Re(|\zeta|_{\mathbb{C}})\frac{\vec{m}}{m_0}.$$

Because $|\vec{m}|/m_0 = \tan\theta \leq \tan\omega$, we obtain the desired equivalence by letting $\vec{m}$ vary over all directions in $\mathbb{R}^n$. □

For each $0 < \omega < \pi/2$, let $S_\omega(\mathbb{C}^n)$ denote the set of all $\zeta \in \mathbb{C}^n$ satisfying condition (8) and let $S^\circ_\omega(\mathbb{C}^n)$ be its interior.

Corollary 3.4. *Let $f : S^\circ_\omega(\mathbb{R}^{n+1}) \to \mathbb{C}_{(n)}$ be a left monogenic function such that the restriction $\tilde{f}$ of f to $\mathbb{R}^n \setminus \{0\}$ takes values in $\mathbb{C}$. Then $\tilde{f}$ is the restriction to $\mathbb{R}^n \setminus \{0\}$ of an analytic function defined on $S^\circ_\omega(\mathbb{C}^n)$*

The sectors $S_\omega(\mathbb{C}^n) \subset \mathbb{C}^n$ and $S_\omega(\mathbb{R}^{n+1}) \subset \mathbb{R}^{n+1}$ are dual to each other in the sense that the mapping

$$(\omega, \zeta) \mapsto G_\omega(\zeta), \quad \omega \in \mathbb{R}^{n+1} \setminus S_\omega(\mathbb{R}^{n+1}),\ \zeta \in S^\circ_\omega(\mathbb{C}^n)$$

is two-sided monogenic in ω and analytic in ζ.

The sector $S_\omega(\mathbb{C}^n)$ arose in [10] as the set of $\zeta \in \mathbb{C}^n$ for which the exponential functions

$$e_+(x,\zeta) = e^{i\langle \vec{x}, \zeta\rangle} e^{-x_0|\zeta|_{\mathbb{C}}} \chi_+(\zeta), \quad x = x_0 e_0 + \vec{x},$$

have decay at infinity for all $x \in \mathbb{R}^{n+1}$ with $\langle x, m\rangle > 0$ and all unit vectors $m = m_0 e_0 + \vec{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 \geq \cot\omega |\vec{m}|$.

4. Joint spectral theory of sectorial multiplication operators

By means of the higher-dimensional analogue (4) of the Riesz-Dunford functional calculus, we can form functions $f(A)$ of a noncommuting system A of operators uniformly of type ω for left monogenic functions f defined on a sector $S_\nu(\mathbb{R}^{n+1})$, $\omega < \nu < \pi/2$, provided that f has decay at zero and infinity.

The observations of the preceding section mean that we can do something similar for the commutative system of multiplication operators in the sector $S_\omega(\mathbb{C}^n)$, although these are not uniformly of type ω. The problem mentioned in the Introduction of connecting monogenic functions defined on a sector in $\mathbb{R}^{n+1}$ with analytic functions defined on a sector in $\mathbb{C}^n$ can be reformulated simply in terms of studying the monogenic functional calculus for multiplication operators.

More precisely, let $0 < \omega < \pi/2$ and set $X = L^2(S_\omega(\mathbb{C}^n), \mathbb{C}_{(n)})$. Integration is with respect to Lebesgue measure on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. Let

M_j be the operator of multiplication by the j'th coordinate function defined on $S_\omega(\mathbb{C}^n)$, that is, the domain $\mathcal{D}(M_j)$ of M_j is the set of all functions $\psi \in X$ such that the function $M_j\psi$ defined by

$$(M_j\psi)(\zeta) = \zeta_j\psi(\zeta), \quad \zeta \in S_\omega(\mathbb{C}^n), \tag{18}$$

belongs to X, and the unbounded operator M_j is given by formula (18) for each $\psi \in \mathcal{D}(M_j)$. Set $M = (M_1, \dots, M_n)$. Then M is a commuting n-tuple of normal operators. Unlike the n-tuple D_Σ of operators mentioned in the Introduction, the existence of a joint functional calculus for M is not an issue.

Indeed, the joint spectral measure $P : \mathcal{B}(S_\omega(\mathbb{C}^n)) \to \mathcal{L}(X)$ given by

$$P(B)\psi = \chi_B.\psi, \quad \psi \in X,\ B \in \mathcal{B}(S_\omega(\mathbb{C}^n)),$$

has support $S_\omega(\mathbb{C}^n)$. For any bounded Borel measurable function $f : S_\omega(\mathbb{C}^n) \to \mathbb{C}$, the bounded linear operator

$$f(M) = \int_{S_\omega(\mathbb{C}^n)} f\, dP$$

is given by the functional calculus for commuting normal operators, and explicitly, by

$$f(M) : \psi \mapsto f.\psi, \quad \psi \in X.$$

Thus we have a functional calculus for M for a class of functions far richer than uniformly bounded analytic functions f defined in a sector $S^\circ_\nu(\mathbb{C}^n)$ with $\omega < \nu < \pi/2$.

Nevertheless, it is not so obvious that bounded linear operators $f(M)$ can also be formed naturally for functions f that are monogenic in a sector $S^\circ_\nu(\mathbb{R}^{n+1})$ with $\omega < \nu < \pi/2$.

The Cauchy kernel $G_x(M)$ is defined for all $x \in N_\nu$ and all ν such that $\omega < \nu < \pi/2$ simply by setting

$$(G_x(M)\psi)(\zeta) = \psi(\zeta)G_x(\zeta), \quad \zeta \in S_\omega(\mathbb{C}^n), \tag{19}$$

for all $\psi \in X$ and $x \in N_\nu$, so that $G_x(M)$ is a left module homomorphism of X. The proof of the next lemma is straightforward and is omitted.

Lemma 4.1. *Let $\omega < \nu < \pi/2$. Then $G_x(M)$ is a bounded linear operator on X for all $x \in N_\nu \setminus \{0\}$. Furthermore, the operator valued function $x \mapsto G_x(M)$, $x \in N_\nu \setminus \{0\}$, is continuous for the strong operator topology, right monogenic in the interior of N_ν and $\|G_x(M)\| = O(|x|^{-n})$ as $|x| \to \infty$ and $|x| \to 0$ in N_ν*

Let $\omega < \nu < \pi/2$. Suppose that f is a left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ such that for every $0 < \theta < \nu$, the bound (7) holds.

Now if $\omega < \theta < \nu$ a nd H_θ is the two-sheeted cone (8) in $\mathbb{R}^{n+1}$, then the bounded linear operator $f(M) : X \to X$ is defined by the formula

$$f(M) = \int_{H_\theta} G_x(M)\boldsymbol{n}(x)f(x)\,d\mu(x) \tag{20}$$

by the same argument by which (9) is defined. Then $f(M)$ is a left module homomorphism of X. The operator $f(M)$ is identified in the next statement.

Proposition 4.2. *Suppose that f is a left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ satisfying the bound (7) for every $0 < \theta < \nu$ and $f(M)$ is defined by formula (20). Let $\tilde{f}$ be the $C_{(n)}$-valued analytic function defined from f by formula (14). Then $\tilde{f}$ is uniformly bounded on the sector $S_\omega(\mathbb{C}^n)$,*

$$(f(M)\psi)(\zeta) = \psi(\zeta)\tilde{f}(\zeta), \quad \zeta \in S_\omega(\mathbb{C}^n), \tag{21}$$

for all $\psi \in X$ and $\|f(M)\| = \|\tilde{f}\|_{H^\infty(S_\omega(\mathbb{C}^n))}$.

The problem remains of obtaining better bounds for the operator norm of $f(M)$ in terms of bounds of the left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ and feeding these bounds back into formula (9) for a system A uniformly of type ω.

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SIMILARITIES OF ω-ACCRETIVE OPERATORS

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Abstract. Given a number $0 < \omega \leq \frac{\pi}{2}$, an ω-accretive operator is a sectorial operator A on Hilbert space whose numerical range lies in the closed sector of all $z \in \mathbb{C}$ such that $|\mathrm{Arg}(z)| \leq \omega$. It is easy to check that any such operator admits bounded imaginary powers, with $\|A^{it}\| \leq e^{\omega|t|}$ for any $t \in \mathbb{R}$. We show that conversely, A is similar to an ω-accretive operator if $\|A^{it}\| \leq e^{\omega|t|}$ for any $t \in \mathbb{R}$.

1. Introduction.

Let H be a Hilbert space and let A be a closed operator on H with dense domain $D(A)$. Given any $\omega \in (0, \pi)$, we let Σ_ω be the open sector of all complex numbers $z \in \mathbb{C}^*$ such that $|\mathrm{Arg}(z)| < \omega$, and we say that A is sectorial of type ω if its spectrum $\sigma(A)$ is included in the closure of Σ_ω and if for every $\theta \in (\omega, \pi)$, the set $\{z(z-A)^{-1} \,:\, z \notin \overline{\Sigma_\theta}\}$ is bounded.

Assume that $\omega \leq \frac{\pi}{2}$. We say that A is ω-accretive if it is sectorial of type ω and if

$$\langle A\xi, \xi\rangle \in \overline{\Sigma_\omega}, \qquad \xi \in D(A). \tag{1.1}$$

It is well-known that if the resolvent set $\rho(A)$ contains -1, say, then (1.1) implies that A is sectorial of type ω. Thus A is ω-accretive if and only if $-1 \in \rho(A)$ and (1.1) holds true. Note that with this terminology, $\frac{\pi}{2}$-accretivity coincides with maximal accretivity. The aim of this note is to give a characterization of injective ω-accretive operators up to similarity in terms of their imaginary powers.

If A is an injective maximal accretive operator on H, then we can define its imaginary powers and we have $\|A^{it}\| \leq e^{\frac{\pi}{2}|t|}$ for any real number $t \in \mathbb{R}$. Indeed this estimate is a consequence of von Neumann's inequality, see e.g. [1, Theorem G]. More generally, assume that A is an injective ω-accretive operator. Then $e^{i(\frac{\pi}{2}-\omega)}A$ and $e^{-i(\frac{\pi}{2}-\omega)}A$ are both maximal accretive hence for any $t \in \mathbb{R}$, we have $\|(e^{i(\frac{\pi}{2}-\omega)}A)^{it}\| \leq e^{\frac{\pi}{2}|t|}$ and $\|(e^{-i(\frac{\pi}{2}-\omega)A})^{it}\| \leq e^{\frac{\pi}{2}|t|}$. We easily deduce that

$$\|A^{it}\| \leq e^{\omega|t|}, \qquad t \in \mathbb{R}. \tag{1.2}$$

Our main result asserts that conversely, if A is an injective sectorial operator satisfying (1.2), then A is similar to an ω-accretive operator, that is, there exists a bounded and invertible operator $S\colon H \to H$ such that $S^{-1}AS$ is ω-accretive. We thus have the following characterization.

Theorem 1.1. *Let $\omega \in (0, \frac{\pi}{2}]$ be a number and let A be an injective sectorial operator on H. Then A is similar to an ω-accretive operator if and only if there exists a bounded and invertible operator $S\colon H \to H$ such that $\|S^{-1}A^{it}S\| \leq e^{\omega|t|}$ for any $t \in \mathbb{R}$.*

We wish to make three comments concerning this theorem. First, it complements a previous result of ours ([7]) saying that if A is an injective sectorial operator of type $< \frac{\pi}{2}$, then A is similar to a maximal accretive operator if and only if it admits bounded imaginary powers. Second, Simard's recent work ([12]) shows that our result is essentially optimal. Indeed on the one hand, [12, Theorem 1] implies that for any $\omega \leq \frac{\pi}{2}$, one can find A not similar to an ω-accretive operator whose imaginary powers satisfy an estimate $\|A^{it}\| \leq Ke^{\omega|t|}$ for some $K > 1$. On the other hand, [12, Theorem 4] shows that one can find A satisfying $\|A^{it}\| \leq e^{\frac{\pi}{2}|t|}$ for any $t \in \mathbb{R}$ without being maximal accretive. The third comment is that our proof heavily relies on some recent work of Crouzeix and Delyon ([5]) who established some remarkable estimates for the analytic functional calculus associated to an operator whose numerical range lies in a band of the complex plane.

We now give a consequence of Theorem 1.1 concerning fractional powers of ω-accretive operators. Let $0 < \omega \leq \frac{\pi}{2}$ and $\alpha \in (0, 1]$ be two numbers. It is well-known that if A is an ω-accretive operator, then A^α is $\alpha\omega$-accretive. Although the converse does not hold true (see e.g. the discussion at the end of [12]), Theorem 1.1 implies the following.

Corollary 1.2. *Let A be an ω-accretive operator for some $\omega \leq \frac{\pi}{2}$ and let $\alpha \geq \frac{2\omega}{\pi}$ be a number. Then $A^{\frac{1}{\alpha}}$ is similar to an $\frac{\omega}{\alpha}$-accretive operator.*

Proof. We may assume that A is injective and that $\alpha \leq 1$. Then our assumption of ω-accretivity implies (1.2). Since $(A^{\frac{1}{\alpha}})^{it} = A^{i\frac{t}{\alpha}}$, we thus have $\|(A^{\frac{1}{\alpha}})^{it}\| \leq e^{\frac{\omega}{\alpha}|t|}$ for any $t \in \mathbb{R}$. According to Theorem 1.1, this implies that $A^{\frac{1}{\alpha}}$ is similar to an $\frac{\omega}{\alpha}$-accretive operator, whence the result by taking α-th powers. $\square$

The proof of Theorem 1.1 is given in Section 3. It uses both H^∞ functional calculus techniques (as introduced by McIntosh in [8]) and a theorem of Paulsen ([9]) reducing our proof to the study of the complete boundedness of an appropriate functional calculus. In Section 2 below, we provide some background on Paulsen's Theorem for the convenience of the reader.

2. Background on complete boundedness and Paulsen's Theorem.

We only give a brief account on complete boundedness and its connections with similarity problems. More information and details, as well as important developments and applications can be found in [10].

Given a Hilbert space H, we let $B(H)$ denote the C^*-algebra of all bounded linear operators on H. If $\mathcal{C}$ is a C^*-algebra and $n \geq 1$ is an integer, we let $M_n(\mathcal{C})$ denote the C^*-algebra of all $n \times n$ matrices with entries in $\mathcal{C}$. Let us describe the resulting norm in two important special cases. Assume first that $\mathcal{C} = B(H)$. Then the C^*-norm on $M_n(B(H))$ is obtained by regarding elements of $M_n(B(H))$ as operators on the Hilbertian direct sum $H \oplus \cdots \oplus H$ of n copies of H. Thus for any $[T_{jk}] \in M_n(B(H))$, we have

$$\big\|[T_{jk}]\big\| = \sup\Big\{\Big(\sum_{j=1}^{n}\Big\|\sum_{k=1}^{n} T_{jk}\xi_k\Big\|^2\Big)^{\frac{1}{2}} \,:\, \xi_k \in H,\ \sum_{k=1}^{n}\|\xi_k\|^2 \leq 1\Big\}. \tag{2.1}$$

Now consider the case when $\mathcal{C} = C_b(\Omega)$ is the space of all bounded and continuous functions $g\colon \Omega \to \mathbb{C}$ on some topological space Ω, equipped with its sup norm. Then the C^*-norm on $M_n(C_b(\Omega))$ is obtained by identifying $M_n(C_b(\Omega))$ with the space $C_b(\Omega; M_n)$ of bounded and continuous functions from Ω into M_n. Thus for any $[g_{jk}] \in M_n(C_b(\Omega))$, we have

$$\big\|[g_{jk}]\big\| = \sup\Big\{\big\|[g_{jk}(\lambda)]\big\|_{M_n} \,:\, \lambda \in \Omega\Big\}. \tag{2.2}$$

Let H be a Hilbert space, let $\mathcal{C}$ be a C^*-algebra and let $E \subset \mathcal{C}$ be a (not necessarily closed) subspace of $\mathcal{C}$. Then the space $M_n(E)$ of $n \times n$ matrices with entries in E may be obviously regarded as embedded in $M_n(\mathcal{C})$. By definition, a linear mapping $u\colon E \to B(H)$ is completely bounded if there exists a constant $K \geq 0$ such that

$$\big\|[u(a_{jk})]\big\|_{M_n(B(H))} \leq K\big\|[a_{jk}]\big\|_{M_n(E)}$$

for any $n \geq 1$ and any $[a_{jk}] \in M_n(E)$. In that case, the least possible K is denoted by $\|u\|_{cb}$ and is called the completely bounded norm of u. If the latter is ≤ 1, then we say that u is completely contractive. Obviously any completely bounded mapping u is bounded, with $\|u\| \leq \|u\|_{cb}$.

Paulsen's Theorem asserts that any completely bounded homomorphism on an operator algebra (= subalgebra of a C^*-algebra) is similar to a completely contractive one. More precisely, we have the following statement (see [9]), that we will use in the situation when $\mathcal{C} = C_b(\Omega)$ for some Ω.

Theorem 2.1. *(Paulsen) Let H be a Hilbert space, let $\mathcal{C}$ be a C^*-algebra, let $\mathcal{A} \subset \mathcal{C}$ be a subalgebra, and consider a linear homomorphism $u\colon \mathcal{A} \to B(H)$. If u is completely bounded, then there exists a bounded invertible operator $S\colon H \to H$ such that the linear homomorphism $u_S\colon \mathcal{A} \to B(H)$ defined by letting $u_S(a) = S^{-1}u(a)S$ for any $a \in \mathcal{A}$ is completely contractive. In particular, $\|S^{-1}u(a)S\| \leq \|a\|$ for any $a \in \mathcal{A}$.*

We finally recall for further use that for any $[\alpha_{jk}] \in M_n$ and for any vectors $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ in a Hilbert space H, we have

$$\left| \sum_{j,k=1}^{n} \alpha_{jk} \langle \xi_k, \eta_j \rangle \right| \leq \big\|[\alpha_{jk}]\big\|_{M_n} \left(\sum_{k=1}^{n} \|\xi_k\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\eta_j\|^2 \right)^{\frac{1}{2}}. \tag{2.3}$$

3. Proof of Theorem 1.1.

We first introduce some notation concerning H^∞ functional calculus associated to sectorial operators (in the sense of [8], [3]). For any $\theta \in (0, \pi)$, we recall that

$$\Sigma_\theta = \{z \in \mathbb{C} \,:\, |\mathrm{Arg}(z)| < \theta\}$$

and we let Γ_θ be the counterclockwise oriented boundary of Σ_θ. Then we let $H_0^\infty(\Sigma_\theta)$ be the space of all bounded analytic functions $f\colon \Sigma_\theta \to \mathbb{C}$ for which there exist two positive numbers $c > 0$, $s > 0$, such that

$$|f(z)| \leq c\, \frac{|z|^s}{1 + |z|^{2s}}, \qquad z \in \Sigma_\theta.$$

We recall that if A is a sectorial operator of type $\omega \in (0, \pi)$ and if $f \in H_0^\infty(\Sigma_\theta)$ for some $\theta \in (\omega, \pi)$, then we may define $f(A) \in B(H)$ by letting

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(z)(z - A)^{-1}\, dz,$$

where $\gamma \in (\omega, \theta)$ is an intermediate angle. (The definition of $f(A)$ does not depend on γ by Cauchy's Theorem.)

We let A be an injective sectorial operator satisfying (1.2) for some $\omega \in (0, \frac{\pi}{2})$ and aim at proving that A is similar to an ω-accretive operator. Recall from [11, Theorem 2] that A is necessarily sectorial of type ω. Changing A into $A^{\frac{\pi}{2\omega}}$, we may assume that $\omega = \frac{\pi}{2}$. We fix some $\theta \in (\frac{\pi}{2}, \pi)$ and we let $\mathcal{A}_0 = H_0^\infty(\Sigma_\theta)$ that we regard (by taking restrictions) as a subalgebra of $C_b(\Sigma_{\frac{\pi}{2}})$. Then we let $\mathcal{A} \subset C_b(\Sigma_{\frac{\pi}{2}})$ be the subalgebra linearly spanned by $\mathcal{A}_0$, the function $f_0(z) = \frac{1}{1+z}$, and the constant function 1. We clearly define a homomorphism $u\colon \mathcal{A} \to B(H)$ by letting $u(f) = f(A)$ for $f \in \mathcal{A}_0$, $u(f_0) = (1+A)^{-1}$, $u(1) = 1$, and then extending linearly. We will prove that

$$u\colon \mathcal{A} \longrightarrow B(H) \quad \text{is completely bounded.} \tag{3.1}$$

Taking this for granted, the conclusion goes as follows. By Paulsen's Theorem, there exists an invertible $S \in B(H)$ such that $\|S^{-1}u(f)S\| \leq \|f\|_{C_b(\Sigma_{\frac{\pi}{2}})}$ for all $f \in \mathcal{A}$. Moreover the function $f(z) = \frac{1-z}{1+z}$ belongs to $\mathcal{A}$ and $u(f) = (1-A)(1+A)^{-1}$. Since we have

$$\|f\|_{C_b(\Sigma_{\frac{\pi}{2}})} = \sup\Bigl\{\Bigl|\frac{1-z}{1+z}\Bigr| \,:\, \mathrm{Re}(z) > 0\Bigr\} = 1,$$

we conclude that

$$S^{-1}(1-A)(1+A)^{-1}S = (1-S^{-1}AS)(1+S^{-1}AS)^{-1} \quad \text{is a contraction.}$$

This shows that $S^{-1}AS$ is maximal accretive.

To prove (3.1), we will change our sectorial functional calculus into a band sectorial functional calculus by means of the Log function. For any $\gamma > 0$, let

$$P_\gamma = \{\lambda \in \mathbb{C} \,:\, |\mathrm{Im}(\lambda)| < \gamma\}$$

and let Δ_γ denote its counterclockwise oriented boundary. Let iB be the generator of the c_0-group $(A^{it})_t$, so that B should be thought as being $\mathrm{Log}(A)$. Our assumption that $\|A^{it}\| \leq e^{\frac{\pi}{2}|t|}$ for any $t \in \mathbb{R}$ means that $iB - \frac{\pi}{2}$ and $-iB - \frac{\pi}{2}$ both generate contractive semigroups on H. Hence $\frac{\pi}{2} - iB$ and $\frac{\pi}{2} + iB$ are both maximal accretive, whence

$$\mathrm{Re}\bigl\langle(\frac{\pi}{2} - iB)\xi, \xi\bigr\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\bigl\langle(\frac{\pi}{2} + iB)\xi, \xi\bigr\rangle \geq 0, \qquad \xi \in D(B).$$

In turn this is equivalent to say that the numerical range of B lies into the closure of $P_{\frac{\pi}{2}}$, that is,

$$\langle B\xi, \xi\rangle \in \overline{P_{\frac{\pi}{2}}}, \qquad \xi \in D(B),\ \|\xi\| \leq 1. \tag{3.2}$$

Let $H_0^\infty(P_\theta)$ be the space of all bounded analytic functions $g\colon P_\theta \to \mathbb{C}$ for which there exist a constant $c > 0$ such that $|g(\lambda)| \leq \frac{c}{1+|\lambda|^2}$ for any $\lambda \in P_\theta$. Then let $\gamma \in (\frac{\pi}{2}, \theta)$ be an arbitrary number. Since $iB - \frac{\pi}{2}$ and $-iB - \frac{\pi}{2}$ both generate contractive semigroups, the function $\lambda \mapsto$

$(\lambda - B)^{-1}$ is well-defined and bounded on Δ_γ hence for any $g \in H_0^\infty(P_\theta)$ we may define $g(B) \in B(H)$ by letting

$$g(B) = \frac{1}{2\pi i} \int_{\Delta_\gamma} g(\lambda)(\lambda - B)^{-1}\, d\lambda.$$

It is easy to check (using Cauchy's Theorem) that this definition does not depend on the choice of γ and that the mapping $v\colon g \mapsto g(B)$ is a linear homomorphism from $H_0^\infty(P_\theta)$ into $B(H)$. Moreover the sectorial and band functional calculi are compatible in the sense that for any $f \in H_0^\infty(\Sigma_\theta)$, the function $\lambda \mapsto f(e^\lambda)$ belongs to $H_0^\infty(P_\theta)$ and

$$g(B) = f(A) \quad \text{if} \quad g(\lambda) = f(e^\lambda). \tag{3.3}$$

We refer the reader to [2] for various relationships between sectorial and band functional calculi, from which a proof of (3.3) can be extracted. However we give a direct argument for the sake of completeness. Let φ be the function defined by $\varphi(z) = z(1+z)^{-2}$, so that $\varphi(A) = A(1+A)^{-2}$. It is well-known that $\varphi(A)$ has a dense range, so that we only need to prove that $g(B)\varphi(A) = f(A)\varphi(A)$. We fix two parameters $\frac{\pi}{2} < \gamma_2 < \gamma_1 < \theta$. Let λ be a complex number with $\mathrm{Im}(\lambda) = \gamma_1$. Applying the Laplace formula to the semigroup $(A^{-it})_{t\geq 0}$, we have (in the strong sense)

$$(\lambda - B)^{-1} = i(i\lambda - iB)^{-1} = -i \int_0^\infty e^{i\lambda t} A^{-it}\, dt.$$

Hence using Fubini's Theorem, we obtain

$$\begin{aligned}(\lambda - B)^{-1}\varphi(A) &= \frac{-1}{2\pi} \int_0^\infty e^{i\lambda t} \int_{\Gamma_{\gamma_2}} z^{-it}\varphi(z)(z - A)^{-1}\, dz\; dt \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\gamma_2}} \left(-i \int_0^\infty e^{i\lambda t} z^{-it}\, dt\right)\varphi(z)(z - A)^{-1}\, dz\end{aligned}$$

whence

$$(\lambda - B)^{-1}\varphi(A) = \frac{1}{2\pi i} \int_{\Gamma_{\gamma_2}} \frac{1}{\lambda - Log(z)}\, \varphi(z)(z - A)^{-1}\, dz.$$

The latter idendity can be proved as well if $\mathrm{Im}(\lambda) = -\gamma_1$ hence holds true for any $\lambda \in \Delta_{\gamma_1}$. Using Fubini's Theorem again and Cauchy's

Theorem, we therefore deduce that

$$\begin{aligned} g(B)\varphi(A) &= \frac{1}{2\pi i}\int_{\Delta_{\gamma_1}} g(\lambda)(\lambda - B)^{-1}\varphi(A)\,d\lambda \\ &= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\Delta_{\gamma_1}} g(\lambda) \int_{\Gamma_{\gamma_2}} \frac{1}{\lambda - Log(z)}\,\varphi(z)(z-A)^{-1}\,dz\,d\lambda \\ &= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\Gamma_{\gamma_2}} \Big(\int_{\Delta_{\gamma_1}} g(\lambda)\frac{1}{\lambda - Log(z)}\,d\lambda\Big)\varphi(z)(z-A)^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_{\gamma_2}} g(Log(z))\varphi(z)(z-A)^{-1}\,dz \\ &= f(A)\varphi(A), \end{aligned}$$

which concludes the proof of (3.3).

We let $\mathcal{B}$ be equal to $H_0^\infty(P_\theta)$ regarded as a subalgebra of $C_b(P_{\frac{\pi}{2}})$. To prove (3.1), it will suffice to show that

$$(3.4) \qquad v\colon \mathcal{B} \longrightarrow B(H) \quad \text{is completely bounded.}$$

Indeed since the exponential function is a holomorphic bijection from $P_{\frac{\pi}{2}}$ onto $\Sigma_{\frac{\pi}{2}}$, it follows from (3.3) and the definition of the matrix norms on $\mathcal{A}$ and $\mathcal{B}$ (see (2.2)) that if v is completely bounded, then $u_{|\mathcal{A}_0}$ is completely bounded as well, with $\|u_{|\mathcal{A}_0}\|_{cb} \leq \|v\|_{cb}$. However $\mathcal{A}_0$ has codimension 2 in $\mathcal{A}$ hence the complete boundednes of $u_{|\mathcal{A}_0}$ implies that of u on $\mathcal{A}$.

We now come to the heart of the proof, which consists in showing that for an operator B whose spectrum is included in $\overline{P_{\frac{\pi}{2}}}$, the condition (3.2) implies (3.4). That (3.2) implies the boundedness of v is a recent result of Crouzeix and Delyon ([5]) and our proof of the complete boundedness of v will essentially be a repetition of their arguments, up to some adequate matrix norm manipulations. Before embarking into computations, we notice that (3.2) is equivalent to the following real/imaginary parts decomposition for B:

$$(3.5) \qquad B = C + iD, \quad \text{with} \quad C = C^*,\ D = D^*,\ \|D\| \leq \frac{\pi}{2}.$$

In this decomposition, C is a possibly unbounded self-adjoint operator with $D(C) = D(B)$. Let $(E(s))_s$ be the resolution of the identity for C and for any integer $m \geq 1$, let

$$C_m = \int_{(-m,m)} s\,dE(s) \qquad \text{and} \qquad B_m = C_m + iD.$$

Then C_m is a bounded self-adjoint operator hence B_m is a bounded operator whose numerical range lies in $\overline{P_{\frac{\pi}{2}}}$. Moreover for any $\lambda \notin \overline{P_{\frac{\pi}{2}}}$, we have

$$(\lambda - B_m)^{-1} \longrightarrow (\lambda - B)^{-1} \quad \text{strongly.} \tag{3.6}$$

Indeed, $(\lambda - B_m)^{-1} - (\lambda - B)^{-1} = (\lambda - B_m)^{-1}(C_m - C)(\lambda - B)^{-1}$, we have $C_m\xi \to C\xi$ for any $\xi \in D(B) = D(C)$, and since the operators $\frac{\pi}{2} \pm iB_m$ are maximal accretive, we have a uniform estimate

$$\|(\lambda - B_m)^{-1}\| \leq d(\lambda, P_{\frac{\pi}{2}}), \qquad m \geq 1. \tag{3.7}$$

Next, by Lebesgue's Theorem, it follows from (3.6) and (3.7) that $g(B_m) \to g(B)$ strongly for any $g \in \mathcal{B}$. Thus for any $n \geq 1$ and any $[g_{jk}] \in M_n(\mathcal{B})$, we have

$$\big\|[g_{jk}(B)]\big\| \leq \limsup_m \big\|[g_{jk}(B_m)]\big\|$$

To prove the complete boundedness of v, it therefore suffices to prove that the mappings $g \mapsto g(B_m)$ are uniformly completely bounded. To achieve this goal we shall now assume that B is bounded and shall prove that

$$\|v\|_{cb} \leq \frac{2}{\sqrt{3}} + 2. \tag{3.8}$$

Let $\gamma \in (\frac{\pi}{2}, \theta)$ be an arbitrary intermediate angle. Then according to [5, (5)] (and its proof), we may write

$$v(g) = g(B) = v_1^\gamma(g) + v_2^\gamma(g)$$

for any $g \in \mathcal{B} = H_0^\infty(P_\theta)$, with

$$v_1^\gamma(g) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} g(x)\big((x + 2\gamma i - B^*)^{-1} - (x - 2\gamma i - B^*)^{-1}\big)\, dx;$$
$$v_2^\gamma(g) = \frac{1}{2\pi i}\int_{\Delta_\gamma} g(\lambda)\big((\lambda - B)^{-1} - (\overline{\lambda} - B^*)^{-1}\big)\, d\lambda.$$

Moreover it is easy to check that for any $x \in \mathbb{R}$, one has

$$\begin{aligned}(x + 2\gamma i - B^*)^{-1} &- (x - 2\gamma i - B^*)^{-1}\\ &= -4\gamma i\,(x + 2\gamma i - B^*)^{-1}(x - 2\gamma i - B^*)^{-1}\\ &= -4\gamma i\,\big(M(x) - iN(x)\big)^{-1},\end{aligned}$$

where $M(x)$ and $N(x)$ are self-adjoint operators defined by

$$M(x) = (x - C)^2 - D^2 + 4\gamma^2 \quad \text{and} \quad N(x) = CD + DC - 2xD.$$

(The boundedness of C allows this real/imaginary parts decomposition.) It follows from (3.5) that

$$M(x) \geq (x-C)^2 + 3\Big(\frac{\pi}{2}\Big)^2. \tag{3.9}$$

In particular, $M(x)$ is invertible and with $Q(x) = M(x)^{-\frac{1}{2}} N(x) M(x)^{-\frac{1}{2}}$, we may write

$$(x+2\gamma i-B^*)^{-1}-(x-2\gamma i-B^*)^{-1} = -4\gamma i\, M(x)^{-\frac{1}{2}}\big(1-iQ(x)\big)^{-1} M(x)^{-\frac{1}{2}}.$$

Let $n \geq 1$ be an integer and let $[g_{jk}]$ be an element of $M_n(\mathcal{B})$ with norm ≤ 1. According to (2.2), this simply means that

$$\big\|[g_{jk}(\lambda)]\big\|_{M_n} \leq 1, \qquad \lambda \in P_{\frac{\pi}{2}}. \tag{3.10}$$

We let $\xi_1, \dots, \xi_n$, $\eta_1, \dots, \eta_n$ be arbitrary elements of H. Then

$$\begin{aligned}
&\sum_{j,k=1}^{n} \big\langle v_1^{\gamma}(g_{jk})\xi_k, \eta_j\big\rangle \\
&= \sum_{j,k=1}^{n} \Big(\frac{-2\gamma}{\pi}\Big) \int_{-\infty}^{+\infty} g_{jk}(x) \big\langle M(x)^{-\frac{1}{2}}\big(1-iQ(x)\big)^{-1} M(x)^{-\frac{1}{2}}\xi_k, \eta_j\big\rangle\, dx \\
&= \Big(\frac{-2\gamma}{\pi}\Big) \int_{-\infty}^{+\infty} \sum_{j,k=1}^{n} g_{jk}(x) \big\langle \big(1-iQ(x)\big)^{-1} M(x)^{-\frac{1}{2}}\xi_k, M(x)^{-\frac{1}{2}}\eta_j\big\rangle\, dx\,.
\end{aligned}$$

Applying (2.3) and (3.10), we obtain that

$$\begin{aligned}
&\Big|\sum_{j,k=1}^{n} \big\langle v_1^{\gamma}(g_{jk})\xi_k, \eta_j\big\rangle\Big| \\
&\leq \frac{2\gamma}{\pi} \int_{-\infty}^{+\infty} \Big(\sum_k \big\|\big(1-iQ(x)\big)^{-1} M(x)^{-\frac{1}{2}}\xi_k\big\|^2\Big)^{\frac{1}{2}} \Big(\sum_j \big\|M(x)^{-\frac{1}{2}}\eta_j\big\|^2\Big)^{\frac{1}{2}} dx.
\end{aligned}$$

Since $Q(x)$ is self-adjoint, the operator $\big(1-iQ(x)\big)^{-1}$ is a contraction for any $x \in \mathbb{R}$ hence applying Cauchy-Schwarz, we finally obtain that

$$\begin{aligned}
&\Big|\sum_{j,k=1}^{n} \big\langle v_1^{\gamma}(g_{jk})\xi_k, \eta_j\big\rangle\Big| \\
&\leq \frac{2\gamma}{\pi} \Big(\sum_k \int_{-\infty}^{+\infty} \big\|M(x)^{-\frac{1}{2}}\xi_k\big\|^2\, dx\Big)^{\frac{1}{2}} \Big(\sum_j \int_{-\infty}^{+\infty} \big\|M(x)^{-\frac{1}{2}}\eta_j\big\|^2\, dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Now observe that for any $\xi \in H$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \left\|M(x)^{-\frac{1}{2}}\xi\right\|^2 dx &= \int_{-\infty}^{+\infty} \langle M(x)^{-1}\xi, \xi\rangle\, dx \\ &\leq \int_{-\infty}^{+\infty} \Big\langle \Big((x-C)^2 + 3\big(\frac{\pi}{2}\big)^2\Big)^{-1}\xi, \xi\Big\rangle\, dx\end{aligned}$$

by (3.9). Moreover using the spectral representation of C we see that the latter integral is equal to

$$\int_{-\infty}^{+\infty} \frac{\|\xi\|^2}{x^2 + 3\big(\frac{\pi}{2}\big)^2}\, dx = \frac{2}{\sqrt{3}}\,\|\xi\|^2.$$

Combining with the previous estimate, this yields

$$\Big|\sum_{j,k=1}^{n} \big\langle v_1^\gamma(g_{jk})\xi_k, \eta_j\big\rangle\Big| \leq \frac{4\gamma}{\pi\sqrt{3}}\Big(\sum_k \|\xi_k\|^2\Big)^{\frac{1}{2}}\Big(\sum_j \|\eta_j\|^2\Big)^{\frac{1}{2}}.$$

In view of the definition of matrix norms on $B(H)$ (see (2.1)), we deduce

$$\big\|[v_1^\gamma(g_{jk})]\big\| \leq \frac{4\gamma}{\pi\sqrt{3}}. \tag{3.11}$$

We now turn to an estimate for v_2^γ. We rewrite the definition of the latter mapping as

$$v_2^\gamma(g) = \int_{\Delta_\gamma} g(\lambda)T(\lambda)\,|d\lambda|,$$

where $T(\lambda)$ is equal to $\frac{1}{2\pi i}\big((\lambda - B)^{-1} - (\overline{\lambda} - B^*)^{-1}\big)$ if $\mathrm{Im}(\lambda) = -\gamma$ and is equal to its opposite if $\mathrm{Im}(\lambda) = \gamma$. The key point is that $T(\lambda)$ is a nonnegative operator for any $\lambda \in \Delta_\gamma$. Indeed assume for example that $\mathrm{Im}(\lambda) = -\gamma$. Then

$$\begin{aligned}&\frac{1}{2\pi i}\big((\lambda - B)^{-1} - (\overline{\lambda} - B^*)^{-1}\big) \\ &\qquad = \frac{1}{2\pi i}(\lambda - B)^{-1}(2i\gamma + B - B^*)(\overline{\lambda} - B^*)^{-1} \\ &\qquad = \frac{1}{\pi}(\lambda - B)^{-1}(\gamma + D)(\overline{\lambda} - B^*)^{-1},\end{aligned}$$

which is nonnegative by (3.5). Then arguing as above, we obtain that for any vectors $\xi_1,\dots,\xi_n$, and $\eta_1,\dots,\eta_n \in H$, we have

$$\Big|\sum_{j,k=1}^{n} \big\langle v_2^\gamma(g_{jk})\xi_k,\eta_j\big\rangle\Big| \leq \sup_{\lambda\in P_\gamma}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}\Big(\sum_k \int_{\Delta_\gamma}\big\|T(\lambda)\xi_k\big\|^2\,|d\lambda|\Big)^{\frac12} \times\Big(\sum_j \int_{\Delta_\gamma}\big\|T(\lambda)\eta_j\big\|^2\,|d\lambda|\Big)^{\frac12}.$$

Now observe that since B is bounded, the function $\lambda\mapsto(\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}$ is integrable on Δ_γ and that $\frac{1}{2\pi i}\int_{\Delta_\gamma}(\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}\,d\lambda = 2$ by Cauchy's Theorem. Hence for any $\xi\in H$, we have

$$\int_{\Delta_\gamma}\big\|T(\lambda)\xi\big\|^2\,|d\lambda| = \frac{1}{2\pi i}\int_{\Delta_\gamma}\big\langle\big((\lambda-B)^{-1}-(\overline{\lambda}-B^*)^{-1}\big)\xi,\xi\big\rangle\,d\lambda = 2\|\xi\|^2.$$

Combining with the above estimate, we obtain that

$$\big\|[v_2^\gamma(g_{jk})]\big\| \leq 2\sup_{\lambda\in P_\gamma}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}.$$

Since

$$\lim_{\gamma\to\frac{\pi}{2}}\Big(\sup_{\lambda\in P_\gamma}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}\Big) = \sup_{\lambda\in P_{\frac{\pi}{2}}}\Big\{\big\|[g_{jk}(\lambda)]\big\|\Big\}\leq 1,$$

we finally deduce that

$$\big\|[v(g_{jk})]\big\| \leq \inf_{\gamma>\frac{\pi}{2}}\Big\{\big\|[v_1^\gamma(g_{jk})]\big\| + \big\|[v_2^\gamma(g_{jk})]\big\|\Big\} \leq \frac{2}{\sqrt{3}}+2,$$

which concludes our proof of (3.8).

Remark 3.1. Two results analogous to the one in [5] appear in [6] and [4]. On the one hand, it is shown in [6] that if $\Omega\subset\mathbb{C}$ is bounded and convex and if B is a bounded operator on H whose numerical range lies in Ω, then the analytic functional calculus associated to B is bounded with respect to the norm induced by $C_b(\Omega)$. On the other hand, it is shown in [4] that if A is an ω-accretive operator on H, then its analytic functional calculus is bounded with respect to the norm induced by $C_b(\Sigma_\omega)$. In the two cases, it it actually possible to show that these bounded functional calculi are completely bounded. If we apply Paulsen's Theorem to the functional calculus considered in [4] (sectorial case), we recover Corollary 1.2.

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JACOBIANS ON LIPSCHITZ DOMAINS OF $\mathbb{R}^2$

ZENGJIAN LOU

Dedicated to Professor Alan McIntosh
on the occasion of his 60th birthday

Abstract. In this note we prove estimates of Jacobian determinants of Du on strongly Lipschitz domains Ω in $\mathbb{R}^2$. The theorem consists of two parts: one is an estimate in terms of the $BMO_r(\Omega)$ norm for u in the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^2)$ with boundary zero, and another is an estimate in terms of the $BMO_z(\Omega)$ norm for u in $W^{1,2}(\Omega, \mathbb{R}^2)$ with no boundary conditions.

1. Introduction

Jacobian determinant estimates were first studied by Müller in [Mu]. In [CLMS], Coifman, Lions, Meyer and Semmes' Theorems II.1 and III.2 imply that for $b \in L^2_{loc}(\mathbb{R}^2)$, $\sup_u \int_{\mathbb{R}^2} b \det Du\, dx$ is equivalent to the $BMO(\mathbb{R}^2)$ norm of b, where $\det Du(x) = \left(\frac{\partial u_j}{\partial x_k}\right)$, the supremum is taken over all u in the Sobolev space $W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \leq 1$. The aim of this note is to consider an extension of this result to domains in $\mathbb{R}^2$. As a main result (Theorem 2.1), we give estimates of $\sup_u \int_\Omega b \det Du\, dx$ when Ω is a strongly Lipschitz domain of $\mathbb{R}^2$, where the superemum is taken over all u in the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^2)$ or $W^{1,2}_0(\Omega, \mathbb{R}^2)$ (the closure of $C^\infty_0(\Omega, \mathbb{R}^2)$ in $W^{1,2}(\Omega, \mathbb{R}^2)$) with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \leq 1$.

In the sequel, Ω will denote a strongly Lipschitz domain - an assumption which is enough to ensure

(1) the existence of a bounded extension map from $W^{1,2}(\Omega, \mathbb{R}^2)$ to $W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$, and

(2) the existence of a bounded extension map from $BMO_r(\Omega)$ to $BMO(\mathbb{R}^2)$, where $BMO_r(\Omega)$ is the space of locally integrable functions with

$$\|f\|_{BMO_r(\Omega)} = \sup_{Q\subset\Omega} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \right) < \infty,$$

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here $f_Q = \frac{1}{|Q|}\int_Q f(x)\, dx$, the supremum is taken over all cubes Q in the domain Ω.

In [CKS], two Hardy spaces are defined on bounded domains Ω, one which is reasonably speaking the largest, and the other which in a sense is the smallest. The largest, $\mathcal{H}^1_r(\Omega)$, arises by restricting to Ω arbitrary elements of $\mathcal{H}^1(\mathbb{R}^2)$. The other, $\mathcal{H}^1_z(\Omega)$, arises by restricting to Ω elements of $\mathcal{H}^1(\mathbb{R}^2)$ which are zero outside $\bar{\Omega}$. Norms on these spaces are defined as following

$$\|f\|_{\mathcal{H}^1_r(\Omega)} = \inf \|F\|_{\mathcal{H}^1(\mathbb{R}^2)},$$

the infimum being taken over all the functions $F \in \mathcal{H}^1(\mathbb{R}^2)$ such that $F|_\Omega = f$,

$$\|f\|_{\mathcal{H}^1_z(\Omega)} = \|F\|_{\mathcal{H}^1(\mathbb{R}^2)},$$

where F is the zero extension of f to $\mathbb{R}^2$.

From [C], the dual of $\mathcal{H}^1_z(\Omega)$ is $BMO_r(\Omega)$ and the dual of $\mathcal{H}^1_r(\Omega)$ is $BMO_z(\Omega)$, where $BMO_z(\Omega)$ is the space of all functions in $BMO(\mathbb{R}^2)$ supported in $\bar{\Omega}$, equipped with the norm $\|f\|_{BMO_z(\Omega)} = \|f\|_{BMO(\mathbb{R}^2)}$.

2. The Main Theorem and Its Proof

In [CLMS, Theorems II.1 and III.2], among other results, Coifman, Lions, Meyer and Semmes established the following:

(A) If $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ then det Du belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and

$$\|\det Du\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C \prod_{i=1}^{2} \|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \tag{2.1}$$

for some absolute constants C.

(B) $b \in L^2_{loc}(\mathbb{R}^2)$

$$\|b\|_{BMO(\mathbb{R}^2)} \sim \sup_{E,F} \int_{\mathbb{R}^2} b\; E \cdot F\; dx, \tag{2.2}$$

where the supremum is taken over all E, $F \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ with div $E =$ curl $F = 0$ and $\|E\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)}$, $\|F\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$.

We will see that (A) and (B) yield the following equivalence

$$\|b\|_{BMO(\mathbb{R}^2)} \sim \sup_{u} \int_{\mathbb{R}^2} b \det Du\; dx, \tag{2.3}$$

where the supremum is taken over all $u = (u_1, u_2) \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1,\ i = 1,\ 2.$

Suppose that E and F satisfy the conditions of (B). From Theorems 2.9 and 3.1 in [GR], there exist $\varphi,\ \psi \in W^{1,2}(\mathbb{R}^2)$ such that

$$E = \text{curl } \varphi = \left(\frac{\partial\varphi}{\partial x_2},\ -\frac{\partial\varphi}{\partial x_1}\right)$$

and

$$F = D\psi = \left(\frac{\partial\psi}{\partial x_1},\ \frac{\partial\psi}{\partial x_2}\right).$$

Define $u = (\varphi, \psi)$. Then $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$, $i = 1,\ 2$, and

$$\det\ Du = -E \cdot F.$$

Thus (2.2) implies that

$$\|b\|_{BMO(\mathbb{R}^2)} \le C \sup_u \left| \int_{\mathbb{R}^2} b \det\ Du\ dx \right|.$$

Conversely, applying (2.1) and the duality $\mathcal{H}^1(\mathbb{R}^2)^* = BMO(\mathbb{R}^2)$, we have

$$\begin{aligned} \textstyle\int_{\mathbb{R}^2} b \det\ Du\ dx &\le C\|b\|_{BMO(\mathbb{R}^2)} \|\det\ Du\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\le C\|b\|_{BMO(\mathbb{R}^2)} \prod_{i=1}^{2} \|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \\ &\le C\|b\|_{BMO(\mathbb{R}^2)} \end{aligned}$$

if $\|Du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$.

A natural question to ask is: under what conditions does (2.3) hold on domains Ω of $\mathbb{R}^2$? As a main theorem of this note, we solve this problem for strongly Lipschitz domains in $\mathbb{R}^2$.

Theorem 2.1. *Let Ω be a strongly Lipschitz domain in $\mathbb{R}^2$.*

(1) If $b \in BMO_z(\Omega)$, then we have equvalence

$$\|b\|_{BMO_z(\Omega)} \sim \sup_u \int_\Omega b \det\ Du\ dx, \tag{2.4}$$

the supremum being taken over all $u = (u_1, u_2) \in W^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1,\ i = 1,\ 2$.

(2) If $b \in BMO_r(\Omega)$, then

$$\|b\|_{BMO_r(\Omega)} \sim \sup_u \int_\Omega b \det\ Du\ dx, \tag{2.5}$$

the supremum being taken over all $u = (u_1, u_2) \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$, $i = 1,\ 2$.

The implicit constants in (2.4) and (2.5) depend only on the domain Ω.

To prove Theorem 2.1, we need the following Lemmas 2.2 - 2.4. The proof of Lemma 2.2 is given at the end of this section. Lemma 2.3 is the two-dimensional case of Theorem 3.1 in Section 3. Lemma 2.4 is a special case of an extension theorem by Jones in [J, Theorem 1]. We also need the the following seminorm defined in [Z]

$$\|b\|_{BMO^H(\Omega)} = \sup_Q \left(\frac{1}{|Q|} \int_Q |b - b_Q| \, dx \right),$$

where the supremum is taken over all cubes Q with $2Q \subset \Omega$.

Lemma 2.2. *Let Ω be an open domain in $\mathbb{R}^2$. For $b \in L^2_{loc}(\Omega)$*

$$\|b\|_{BMO^H(\Omega)} \leq C \sup_u \int_\Omega b \det Du \, dx$$

for a constant C independent of b, the supremum being taken over all $u = (u_1, u_2) \in W^{1,2}_0(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \leq 1$, $i = 1,\ 2$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a strongly Lipschitz domain and let b be a locally integrable function on Ω. Then*

$$\|b\|_{BMO_r(\Omega)} \sim \|b\|_{BMO^H(\Omega)},$$

where the implicit constants are independent of b.

Lemma 2.4. *Let Ω be a strongly Lipschitz domain in $\mathbb{R}^2$ and let $b \in BMO_r(\Omega)$. Then there exists $B \in BMO(\mathbb{R}^2)$ such that*

$$B|_\Omega = b$$

and

$$\|B\|_{BMO(\mathbb{R}^2)} \leq C\|b\|_{BMO_r(\Omega)}$$

for some constants C independent of B and b.

Proof of Theorem 2.1. (1) Suppose $b \in BMO_z(\Omega)$. Define

$$B = \begin{cases} b & \text{in } \Omega; \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega. \end{cases}$$

By the definition of $BMO_z(\Omega)$, $B \in BMO(\mathbb{R}^2)$ and

$$\|B\|_{BMO(\mathbb{R}^2)} = \|b\|_{BMO_z(\Omega)}. \tag{2.6}$$

Since Ω is a bounded strongly Lipschitz domain, $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ can be extended to $U \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ with

$$\|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \leq C\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)}, \tag{2.7}$$

where the constant C depends only on the Lipschitz constant of Ω (see, for example, Proposition 4.12 in [HLMZ]). Therefore (2.6), (2.7) and (2.1) give

$$\begin{aligned}\int_\Omega b \det Du\, dx &= \int_{\mathbb{R}^2} B \det DU\, dx \\ &\le \|B\|_{BMO(\mathbb{R}^2)} \|\det DU\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\le C\|b\|_{BMO_z(\Omega)} \prod_{i=1}^2 \|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \\ &\le C\|b\|_{BMO_z(\Omega)} \prod_{i=1}^2 \|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \\ &\le C\|b\|_{BMO_z(\Omega)}\end{aligned}$$

if $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$, where C depends only on the domain Ω.

We now prove the converse. Let $b \in BMO_z(\Omega)$ and define B as above. Suppose $U \in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2)$ and $\|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1$. Let $u = U|_\Omega$, then $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$. So (2.3) and (2.6) yield

$$\begin{aligned}\|b\|_{BMO_z(\Omega)} &= \|B\|_{BMO(\mathbb{R}^2)} \\ &\le C \sup_{U\in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2),\|DU_i\|_{L^2}\le 1} \int_{\mathbb{R}^2} B \det DU\, dx \\ &= C \sup_{u=U|_\Omega, U\in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2),\|DU_i\|_{L^2}\le 1} \int_\Omega b \det Du\, dx \\ &\le C \sup_{u\in W^{1,2}(\Omega,\mathbb{R}^2),\|Du_i\|_{L^2}\le 1} \int_\Omega b \det Du\, dx.\end{aligned}$$

(2) Let $B \in BMO(\mathbb{R}^2)$ is an extension of $b \in BMO_r(\Omega)$. For $u \in W_0^{1,2}(\Omega,\mathbb{R}^2)$, define

$$U = \begin{cases} u & \text{in } \Omega; \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Then $U \in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2)$ with

$$\|U\|_{W^{1,2}(\mathbb{R}^2,\mathbb{R}^2)} = \|u\|_{W^{1,2}(\Omega,\mathbb{R}^2)}.$$

By duality $\mathcal{H}^1(\mathbb{R}^2)^* = BMO(\mathbb{R}^2)$, (2.1) and Lemma 2.4, we have

$$\begin{aligned}
\int_\Omega b \det Du\, dx &= \int_{\mathbb{R}^2} B \det DU\, dx \\
&= \|B\|_{BMO(\mathbb{R}^2)} \|\det DU\|_{\mathcal{H}^1(\mathbb{R}^2)} \\
&\le C\|b\|_{BMO_r(\Omega)} \prod_{i=1}^{2} \|DU_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \\
&= C\|b\|_{BMO_r(\Omega)} \prod_{i=1}^{2} \|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \\
&\le C\|b\|_{BMO_r(\Omega)}
\end{aligned}$$

for all $u \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$, $i = 1, 2$, where the constant C depends only on the domain Ω.

The proof of the reversed inequality in (2.5) follows from Lemmas 2.2 and 2.3. Theorem 2.1 is proved. $\square$

We now prove Lemma 2.2, its proof uses the following result of Nečas [N, Lemma 7.1, Chapter 3].

Lemma 2.5. *Let Ω be a bounded strongly Lipschitz domain in $\mathbb{R}^N$. Then the divergence operator is a (continuous) map from $W_0^{1,2}(\Omega, \mathbb{R}^N)$ onto $L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f\, dx = 0\}$. That is, there exists a constant C depending only on the domain Ω and the dimension N such that for any $f \in L_0^2(\Omega)$, there exists $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that*

$$f = \operatorname{div} \varphi$$

and

$$\|D\varphi\|_{L^2(\Omega,\mathbb{R}^N)} \le C\|f\|_{L^2(\Omega)}.$$

Proof of Lemma 2.2. Suppose $b \in L^2_{loc}(\Omega)$. We will show that there exists $u = (u_1, u_2) \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with $\|Du_i\|_{L^2(\Omega,\mathbb{R}^2)} \le 1$ for $i = 1,\ 2$, and supp $(\det Du) \subset Q$ such that for all cubes Q with $2Q \subset \Omega$,

$$\left(\frac{1}{|Q|}\int_Q |b - b_Q|^2\, dx\right)^{1/2} \le C\left|\int_Q b \det Du\, dx\right|, \tag{2.8}$$

where $b_Q = \frac{1}{|Q|}\int_Q b\, dx$, C is a constant independent of Q, b and u.

Let $h = b - b_Q$, then $h \in L^2(Q)$ with $\int_Q h\, dx = 0$. Using Lemma 2.5 with $\Omega = Q$, there exists $\varphi = (\varphi_1, \varphi_2) \in W_0^{1,2}(Q, \mathbb{R}^2)$ and an absolute constant C_0 such that

$$h = \operatorname{div} \varphi$$

and

$$\|D\varphi\|_{L^2(Q,\mathbb{R}^2)} \leq C_0\|h\|_{L^2(Q)}. \tag{2.9}$$

So

$$\begin{aligned}
\|h\|^2_{L^2(Q)} &= \int_Q h \text{ div } \varphi \; dx \\
&= \int_Q h\frac{\partial \varphi_1}{\partial x_1} \; dx + \int_Q h\frac{\partial \varphi_2}{\partial x_2} \; dx \\
&\leq 2 \max_{1\leq i\leq 2}\Big| \int_Q h\frac{\partial \varphi_i}{\partial x_i} \; dx\Big| \\
&= 2\Big| \int_Q h\frac{\partial \varphi_{i_0}}{\partial x_{i_0}} \; dx\Big|
\end{aligned} \tag{2.10}$$

for some choice of i_0 ($i_0 = 1$ or 2).

Assuming without loss of generality that $i_0 = 1$ in (2.10). To prove (2.8), we need only to show that there exists $u \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ with conditions stated above and a constant C (independent of Q, φ and u) such that

$$\Big| \int_Q h\|h\|^{-1}_{L^2(Q)} \frac{\partial \varphi_1}{\partial x_1} \; dx\Big| \leq C|Q|^{1/2}\Big| \int_Q h \text{ det } Du \; dx\Big|. \tag{2.11}$$

Set $u_1 = \frac{\varphi_1}{C_0\|h\|_{L^2(Q)}}$. It is obvious that $u_1 \in W_0^{1,2}(Q)$ and $\|Du_1\|_{L^2(Q,\mathbb{R}^2)} \leq 1$ by (2.9).

Let $\psi_0 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\psi_0 = \begin{cases} 1 & \text{on } [-1,1]^2; \\ 0 & \text{outside } [-2,2]^2. \end{cases}$$

Define

$$u_2 = \gamma C_0 |Q|^{-1/2}(x_2 - x_2^0)\psi_Q(x),$$

where $\psi_Q(x) = \psi_0\left(\frac{x-x^0}{l(Q)/2}\right)$, $x^0 = (x_1^0, x_2^0)$ denotes the center of the cube Q, $\gamma > 0$ is a normalization constant (independent of x^0 and $l(Q)$) so that $\|Du_2\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \leq 1$. It is obvious that $u_2 \in C_0^\infty(2Q)$.

Let $u = (u_1, u_2)$. By a simple computation, we get

$$\text{det } Du = \gamma |Q|^{-1/2}\|h\|^{-1}_{L^2(Q)}\frac{\partial \varphi_1}{\partial x_1} \quad \text{in} \quad Q.$$

So (2.11) follows. Lemma 2.2 is proved. □

3. The Equivalence of Two BMO Seminorms

In [Z] Zhang asked if the two seminorms $\|b\|_{BMO_r(\Omega)}$ and $\|b\|_{BMO^H(\Omega)}$ are equivalent under suitable conditions on domains Ω in $\mathbb{R}^N$. The following theorem gives a positive answer. No smoothness conditions are needed on the domains. In addition, we will see that the equivalence of $\|b\|_{BMO_r(\Omega)}$ and $\|b\|_{BMO^H(\Omega)}$ implies that $\mathcal{H}^1_z(\Omega)$ can be decomposed into a sum of atoms with supports away from boundaries of the domains (Proposition 3.2).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a strongly Lipschitz domain and let b be a locally integrable function on Ω. Then*

$$\|b\|_{BMO_r(\Omega)} \sim \|b\|_{BMO^H(\Omega)},$$

where the implicit constants are independent of the function b.

Proof. It is obvious that $\|b\|_{BMO^H(\Omega)} \leq \|b\|_{BMO_r(\Omega)}$. Now we prove

$$\|b\|_{BMO_r(\Omega)} \leq C\|b\|_{BMO^H(\Omega)}.$$

We will only give the proof for the case of $\mathbb{R}^N$ for $N = 2$ using ideas of Jones [J]. The case for $\mathbb{R}^N$ ($N \neq 2$) is similar. In fact, from Jones' extension theorem [J, Theorem 1] we need only to prove that there exists a constant C independent of Q and b such that for all cubes $Q \subset \Omega$,

$$\frac{1}{|Q|}\int_Q |b - b_Q|\ dx \leq C\|b\|_{BMO^H(\Omega)}. \tag{3.1}$$

Let $E = \{Q_k\}$ be the dyadic Whitney decomposition of Q, then $Q = \bigcup_k Q_k$ and

$$Q_j \cap Q_k = \emptyset, \quad j \neq k; \tag{3.2}$$

$$1 \leq \frac{d(Q_k, Q^c)}{l(Q_k)} \leq 4 \cdot 2^{1/2}; \tag{3.3}$$

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4 \quad \text{if} \quad Q_j \cap Q_k \neq \emptyset \tag{3.4}$$

(see, for example, Stein's book [S1, page 167] for more information on Whitney decompositions). For $m = 1, 2, ...$, let

$$A_m = \left\{x \in Q : 2^{-m} \leq \frac{d(x, Q^c)}{l(Q)} < 2^{-m+1}\right\}$$

and

$$F_m = \{Q_j \in E : Q_j \cap A_m \neq \emptyset\}.$$

To prove (3.1), we need the following two claims:

Claim (A).

$$\sum_{Q_j \in F_m} |Q_j| \le 40 \cdot 2^{-m} |Q|$$

for all $m = 1, 2, \dots$.

Claim (B). *If* $b \in BMO^H(\Omega)$, *then*

$$|b_{Q_j} - b_{Q_0}| \le Cm\|b\|_{BMO^H(\Omega)}$$

for all $Q_j \in F_m$, $m = 1, 2, \dots$, *where* Q_0 *be the Whitney cube that contains the center of* Q *and* C *is a constant independent of* b *and* Q_j.

We shall give the proofs of these claims later on and we first prove (3.1) admitting them. Since $Q_j \in E$, $j = 1, 2, \dots$, it is obvious that

$$\frac{1}{|Q_j|} \int_{Q_j} |b - b_{Q_j}| \, dx \le \|b\|_{BMO^H(\Omega)}. \tag{3.5}$$

By (3.5), Claims (A) and (B), we have

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |b - b_{Q_0}| \, dx \le & \sum_{m=1}^{\infty} \sum_{Q_j \in F_m} \frac{1}{|Q|} \int_{Q_j} |b - b_{Q_0}| \, dx \\
= & \sum_{m=1}^{\infty} \sum_{Q_j \in F_m} \frac{|Q_j|}{|Q|} \left(|b_{Q_j} - b_{Q_0}| + \frac{1}{|Q_j|} \int_{Q_j} |b - b_{Q_j}| \, dx \right) \\
\le & \sum_{m=1}^{\infty} \sum_{Q_j \in F_m} \frac{|Q_j|}{|Q|} (Cm\|b\|_{BMO^H(\Omega)} + \|b\|_{BMO^H(\Omega)}) \\
\le & \sum_{m=1}^{\infty} 40 \cdot 2^{-m} (Cm + 1) \|b\|_{BMO^H(\Omega)} \\
\le & C\|b\|_{BMO^H(\Omega)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |b - b_Q| \, dx &\le \frac{1}{|Q|} \int_Q (|b - b_{Q_0}| + |b_{Q_0} - b_Q|) \, dx \\
&\le |b_{Q_0} - b_Q| + \frac{1}{|Q|} \int_Q |b - b_{Q_0}| \, dx \\
&\le \frac{2}{|Q|} \int_Q |b - b_{Q_0}| \, dx \\
&\le C\|b\|_{BMO^H(\Omega)}.
\end{aligned}$$

This gives (3.1). $\square$

The proof of claims (A) was given in [J, page 47 (3.8)]. To prove Claim (B) we need the following

Claim (C). *If $b \in BMO^H(\Omega)$ and Q_j, $Q_k \in E$ have touching edges for $j \neq k$, then*

$$|b_{Q_j} - b_{Q_k}| \leq C\|b\|_{BMO^H(\Omega)}$$

for an absolute constant C.

Proof. Suppose that Q_j and Q_k touch and satisfy (3.4). Dividing (3.4) into two cases, case (a):

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 1 \tag{3.6}$$

and case (b):

$$1 < \frac{l(Q_j)}{l(Q_k)} \leq 4. \tag{3.7}$$

For case (a), constructing cubes R_j, R_k and R_{jk} such that

1) $R_j \subset Q_j$, $R_k \subset Q_k$, $l(R_j) = \frac{1}{2}l(Q_j)$, $l(R_k) = \frac{1}{2}l(Q_k)$ and R_j, R_k touch;

2) $R_j,\ R_k \subset R_{jk}$, $l(R_{jk}) = l(R_j) + l(R_k)$;

3) $l(R_{jk}) \leq d(R_{jk}, Q^c)$.

Since $Q_j,\ Q_k \in E$ touch and $Q_j \cap Q_k = \emptyset$, it is easy to find cubes R_j, R_k and R_{jk} satisfying 1) and 2). In order for the cube R_{jk} to satisfy 3) we need only to choose R_{jk} such that $d(Q_j, Q^c) \leq d(R_{jk}, Q^c)$ and $d(Q_k, Q^c) \leq d(R_{jk}, Q^c)$. Therefore

$$\begin{aligned}
l(R_{jk}) &= \frac{1}{2}l(Q_j) + \frac{1}{2}l(Q_k) \\
&\leq \frac{1}{2}\Big(d(Q_j, Q^c) + \frac{1}{2}d(Q_k, Q^c)\Big) \\
&\leq d(R_{jk}, Q^c).
\end{aligned}$$

From 1) and (3.5) we have

$$\begin{aligned}
|b_{R_j} - b_{Q_j}| &= \frac{1}{|R_j|}\Big|\int_{R_j} (b - b_{Q_j})\ dx\Big| \\
&\leq \frac{4}{|Q_j|}\int_{Q_j} |b - b_{Q_j}|\ dx \\
&\leq 4\|b\|_{BMO^H(\Omega)}.
\end{aligned} \tag{3.8}$$

Similarly we get

$$|b_{R_k} - b_{Q_k}| \leq 4\|b\|_{BMO^H(\Omega)}. \tag{3.9}$$

By (3.6), 1) and 2) we know that

$$\begin{aligned} l(R_{jk}) &= \frac{l(Q_j)}{2} + \frac{l(Q_k)}{2} \\ &\le \frac{5}{2} l(Q_j) = 5l(R_j) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} l(R_{jk}) &= \frac{l(Q_j)}{2} + \frac{l(Q_k)}{2} \\ &\le l(Q_k) = 2l(R_k). \end{aligned} \tag{3.11}$$

Note that $R_{jk} \subset \Omega$ and $l(R_{jk}) \le d(R_{jk}, \Omega^c)$. Then (3.10) yields

$$\begin{aligned} |b_{R_j} - b_{R_{jk}}| &= \frac{1}{|R_j|} \Big| \int_{R_j} (b - b_{R_{jk}})\, dx \Big| \\ &\le \frac{25}{|R_{jk}|} \int_{R_{jk}} |b - b_{R_{jk}}|\, dx \\ &\le 25 \|b\|_{BMO^H(\Omega)}. \end{aligned} \tag{3.12}$$

By (3.11), similar to (3.12) we have

$$|b_{R_k} - b_{R_{jk}}| \le 4 \|b\|_{BMO^H(\Omega)}. \tag{3.13}$$

Combining (3.8), (3.9), (3.12) and (3.13) we get

$$\begin{aligned} |b_{Q_j} - b_{Q_k}| &\le |b_{Q_j} - b_{R_j}| + |b_{R_j} - b_{R_{jk}}| + |b_{R_{jk}} - b_{R_k}| + |b_{R_k} - b_{Q_k}| \\ &\le 37 \|b\|_{BMO^H(\Omega)} \end{aligned}$$

for all Q_j, Q_k satisfying (3.6).

For case (b), repeat the process above we obtain

$$|b_{Q_j} - b_{Q_k}| \le 37 \|b\|_{BMO^H(\Omega)}$$

for all Q_j, Q_k satisfying (3.7).

Therefore for all Q_j, $Q_k \in E$ $(j \ne k)$ have touching edges

$$|b_{Q_j} - b_{Q_k}| \le 37 \|b\|_{BMO^H(\Omega)}.$$

This proves Claim C. □

Proof of Claim (B). The proof is similar to the argument in [J]. Let $x_j \in Q_j \in F_m$, x_Q be the center of the cube Q. Then (3.3) and (3.4) show that there are at most 50 cubes $Q_k \in F_m$ intersect the line segment $\overline{x_j x_Q}$ and at most m sets A_i, $i = 1, 2, ..., m$, intersect $\overline{x_j x_Q}$. So from Claim (C), we have

$$|b_{Q_j} - b_{Q_0}| \le Cm \|b\|_{BMO^H(\Omega)}.$$

Claim (B) is proved. The proof of Theorem 3.1 is finished completely. □

Remark. It should be added that at the time Theorem 3.1 was finished, the author was unfortunately unaware of a similar work in [RR] (with a different proof). Thanks go to P. Schvartsman (Department of Mathematics, Techion, 3200 Haifa, Israel) for pointing this out to him. In addition, Auscher and Russ also proved Theorem 3.1 by using duality [AR, Theorem 6].

We know that any f in $\mathcal{H}^1_z(\Omega)$ has a decomposition (see [CKS], [C] and [JSW] for bounded domains, [AR] for unbounded domains)

$$f = \sum_{k=0}^{\infty} \lambda_k a_k$$

with $\sum_k |\lambda_k| \le C\|f\|_{\mathcal{H}^1_z(\Omega)}$, where the a_k's are $\mathcal{H}^1_z(\Omega)$-atoms: there exist cubes $Q_k \subset \Omega$ such that supp $a_k \subset Q_k$, $\int_{Q_k} a_k \, dx = 0$ and $\|a_k\|_{L^2(Q_k)} \le |Q_k|^{-1/2}$. In the following proposition we prove that the supports of these atoms can be away from the boundary of Ω by using Theorem 3.1. Let $\mathcal{H}^1_{z,2at}(\Omega)$ denote the space of $f \in \mathcal{H}^1_z(\Omega)$ which can be decomposed into a sum of $\mathcal{H}^1_z(\Omega)$-atoms supported in cubes Q with $2Q \subset \Omega$.

Proposition 3.2. *For a strongly Lipschitz domain Ω in $\mathbb{R}^N$*

$$\mathcal{H}^1_z(\Omega) = \mathcal{H}^1_{z,2at}(\Omega).$$

Proof. Obviously we have that $\mathcal{H}^1_{z,2at}(\Omega) \subset \mathcal{H}^1_z(\Omega)$. So to prove the proposition we only need to show that $\mathcal{H}^1_{z,2at}(\Omega)^* \subset BMO_r(\Omega) = BMO^H(\Omega) := \{f : \|f\|_{BMO^H(\Omega)} < \infty\}$. Suppose that L is a bounded linear functional on $\mathcal{H}^1_{z,2at}(\Omega)$. Follow the lines of the proof of Theorem 1 in [S2, Chapter IV] or Theorem 2.5 in [GHL], we see that there exists a function $g \in BMO^H(\Omega)$ such that

$$L(f) = \int_\Omega f(x)g(x)\, dx$$

for all $f \in \mathcal{H}^1_{z,2at}(\Omega)$. The proof is finished. □

4. An Application

In this section we give an application of Theorem 3.1 which improves a coercivity result by Zhang. In [Z], Zhang studied the coercivity of strongly elliptic quadratic forms with measurable coefficients, defined on a bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary,

$$a(u, \Omega) = \int_\Omega A^{ij}_{\alpha,\beta}(x) \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta}\, dx, \quad u \in W^{1,2}_0(\Omega, \mathbb{R}^2),$$

where $A^{ij}_{\alpha,\beta} \in L^\infty(\Omega)$ and satisfy Legendre-Hadamard condition

$$A^{ij}_{\alpha,\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geq c|\xi|^2|\eta|^2.$$

From [M], $A^{ij}_{\alpha,\beta}P^i_\alpha P^j_\beta$ can be written in the form

$$A^{ij}_{\alpha,\beta}P^i_\alpha P^j_\beta = B^{ij}_{\alpha,\beta}P^i_\alpha P^j_\beta + b(x) \det P, \tag{4.1}$$

where $P \in M^{2\times 2}$, the set of real-valued 2×2 matrices, $B^{ij}_{\alpha,\beta} \in L^\infty(\Omega)$ and satisfying

$$C_1|P|^2 \leq B^{ij}_{\alpha,\beta}(x)P^i_\alpha P^j_\beta \leq C_2|P|^2, \tag{4.2}$$

for constants $C_1,\ C_2 > 0$.

As one of the main results in [Z], the following theorem was proved by Zhang which establishes the necessary condition such that $a(u,\Omega) \geq 0$.

Theorem 4.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a strongly Lipschitz domain, $A^{ij}_{\alpha,\beta} : \Omega \to \mathbb{R}^2$ is measurable for $1 \leq i,j,\alpha,\beta \leq 2$, such that (4.1) holds, where $b \in BMO_r(\Omega)$ and $B^{ij}_{\alpha,\beta}$ are measurable functions satisfying (4.2) for constants $0 < C_1 < C_2$. Then there exists a constant C_3 depending only on C_2 such that $a(u,\Omega) \geq 0$ for all $u \in W^{1,2}_0(\Omega,\mathbb{R}^2)$ implies that $\|b\|_{BMO^H(\Omega)} \leq C_3$.*

Theorem 4.1 tells us that if $a(u,\Omega) \geq 0$ for all $u \in W^{1,2}_0(\Omega,\mathbb{R}^2)$, then $\|b\|_{BMO^H(\Omega)} \leq C$, that is, $\|b\|_{BMO_r(\Omega)} \leq C$ by Theorem 3.1. From the Remark in [Z, page 426], if $\|b\|_{BMO_r(\Omega)}$ is sufficient small, then $a(u,\Omega) \geq 0$. We see that $\|b\|_{BMO_r(\Omega)} \leq C$ is "almost" a necessary and sufficient condition of $a(u,\Omega) \geq 0$ for all $u \in W^{1,2}_0(\Omega,\mathbb{R}^2)$.

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DIVERGENT SUMS OF SPHERICAL HARMONICS

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Abstract. We combine the Cantor-Lebesgue Theorem and Uniform Boundedness Principle to prove a divergence result for Cesàro and Bochner-Riesz means of spherical harmonic expansions.

1. Background

Fix an integer $d > 1$ and consider the unit sphere S^d in $\mathbb{R}^{d+1}$, equipped with normalized rotation-invariant measure. For each $n \geq 0$ let $\mathcal{H}_n$ denote the space of spherical harmonics of degree n restricted to S^d, so that $L^2(S^d) = \oplus_{n=0}^{\infty} \mathcal{H}_n$. See [22, Section 4.2] for details. Every distribution ψ on S^d has a spherical harmonic expansion

$$\sum_{n=0}^{\infty} Y_n(\psi)(x), \qquad \forall x \in S^d, \text{ where } Y_n(\psi) \in \mathcal{H}_n, \quad \forall n \geq 0. \tag{1}$$

This is the expansion of ψ in eigenfunctions of the Laplace-Beltrami operator on S^d. It is known [14] that if $1 \leq p < 2$ then there is an $\psi \in L^p(S^d)$ for which (1) diverges almost everywhere. That leaves open the general behaviour of spherical harmonic expansions for elements of $L^2(S^d)$. A partial step in this direction follows from the localization principle [18].

Theorem 1.1 (Localization). *Suppose ψ is a distribution on S^d and $U \subset S^d$ is an open set disjoint from the support of ψ. For each $x \in U$, the expansion $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges if and only if $Y_n(\psi)(x) \to 0$ as $n \to \infty$.*

Corollary 1.2. *If $\psi \in L^2(S^d)$ and $U \subset S^d$ is an open set on which ψ is zero almost everywhere, then the expansion $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges to zero almost everywhere on U.*

There are special cases where a function $\psi \in L^2(S^d)$ can be guaranteed to have an almost everywhere convergent spherical harmonic expansion, if ψ is in an L^2- Sobolev space $W^{2,s}$ of positive index s [16] or if it is zonal [1]. (Recall that a function f on S^d is said to be *zonal* about a point $y \in S^d$ when $f(x)$ depends only on $x \cdot y$ for all $x \in S^d$.)

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Carleson's theorem [3] has been extended to zonal functions [11]. Let p_c be the critical index

$$p_c = \frac{2d}{d+1}.$$

Theorem 1.3. *If $p_c < p \le 2$ and $f \in L^p(S^d)$ is zonal about a point $y \in S^d$, then its spherical harmonic expansion is convergent almost everywhere.*

Corollary 1.4. *Suppose $\psi \in L^2(S^d)$, $U \subset S^d$ is an open set, $f_1 \in \bigcup_{s>0} W^{2,s}(S^d)$, $f_2 \in L^2(S^d)$ is a finite sum of zonal functions, and $\psi = f_1 + f_2$ almost everywhere on U. Then $\sum_{n=0}^{\infty} Y_n(\psi)(x)$ converges almost everywhere on U.*

The two corollaries 1.2 and 1.4 would be rendered trivial if there where a higher dimensional version of Carleson's theorem.

They do suggest that when considering convergence of expansions, we should examine the term-wise behaviour away from the support of a distribution.

In the early 1980's we showed [17] that Theorem 1.3 is sharp and that localization fails at the critical index.

Theorem 1.5. *For each $y \in S^d$ and $1 \le p \le p_c$ there is a $\psi \in L^p(S^d)$, supported in the hemisphere $\{x : x \cdot y \ge 0\}$ whose spherical harmonic expansion diverges almost everywhere.*

This was proved by a combination of the Cantor-Lebesgue theorem, knowledge of the $L^{p'}$-norms of the zonal spherical functions, and the uniform boundedness principle. Kanjin [13] showed that these methods could be combined with a result of Hardy and Riesz [12] to deal with Riesz means for radial functions on Euclidean space. This approach was also used in [20] for Riesz means of radial functions on non-compact rank one symmetric spaces.

Here we prove a similar result for Cesàro and Riesz means of spherical harmonic expansions of zonal functions. This shows the sharpness of the results in [4]. See [2, 7] for earlier work on Cesàro means of spherical harmonic expansions. See [21, 5] for results in a more general setting.

2. Cesàro & Riesz means

2.1. **Cesàro means.** The Cesàro means [24, pages 76–77] of order δ of the expansion (1) are defined by

$$\sigma_N^\delta \psi(x) = \sum_{n=0}^{N} \frac{A_{N-n}^\delta}{A_N^\delta} Y_n(\psi)(x), \qquad \forall N \ge 0, x \in S^d, \tag{2}$$

where $A_n^\delta = \binom{n+\delta}{n}$. Theorem 3.1.22 in [24] says that if the Cesàro means converge, then the terms of the series have controlled growth.

Lemma 2.1. *Suppose that* $\lim_{N\to\infty} \sigma_N^\delta \psi(x)$ *exists for some* $x \in X$ *and* $\delta > -1$. *Then*

$$|Y_N(\psi)(x)| \le C_\delta\, N^\delta \max_{0\le n\le N} \left|\sigma_n^\delta \psi(x)\right|, \qquad \forall n \ge 0.$$

2.2. **Riesz means.** Hardy and Riesz [12] had proved a similar result for Riesz means. Recall that the Riesz means of order $\delta \ge 0$ are defined for each $r > 0$ by

$$S_r^\delta \psi(x) = \sum_{0\le n<r} \left(1 - \frac{k}{r}\right)^\delta Y_n(\psi)(x). \tag{3}$$

Theorem 21 of [12] tells us how the convergence of $S_r^\delta \psi(x)$ controls the size of the partial sums $S_r^0 \psi(x)$.

Lemma 2.2. *Suppose that* ψ *is a distribution on the sphere for which there is some* $\delta > 0$ *and* $x \in X$ *at which its Riesz means* $S_r^\delta \psi(x)$ *converges to* c *as* $r \to \infty$ *then*

$$\left|S_r^0 \psi(x) - c\right| \le A_\delta\, r^\delta \sup_{0<t\le r+1} \left|S_t^\delta \psi(x)\right|.$$

Note that this implies

$$Y_n(\psi)(x) = \mathbf{O}(n^\delta)$$

and we have the same growth estimates as in Lemma 2.1.

Gergen[9] wrote formulae relating the Riesz and Cesàro means of order $\delta \ge 0$, from which it follows that the two methods of summation are equivalent.

3. Zonal Functions and Jacobi Polynomials

3.1. **Notation.** Suppose that f is a function on S^d with $f(x)$ depending only on $x\cdot y$, for a fixed $y \in S^d$, so that $f(x) = f_0(x \cdot y)$. The spherical harmonic expansion of f is

$$\sum_{n=0}^{\infty} c_n(f_0) h_n^{-1} P_n^{(\alpha,\alpha)}(x \cdot y) \tag{4}$$

where $\alpha = (d-2)/2$, $P_n^{(\alpha,\alpha)}$ is the Jacobi polynomial of degree n and index (α, α),

$$h_n = \int_{-1}^{1} \left|P_n^{(\alpha,\alpha)}(t)\right|^2 \left(1-t^2\right)^\alpha dt,$$

and the coefficients are

$$c_n(f_0) = \int_{-1}^{1} f_0(t) P_n^{(\alpha,\alpha)}(t) \left(1-t^2\right)^\alpha dt, \quad \forall n \ge 0.$$

See section 4.7 of Szegő's book [23] for details about these special functions. Let m_α be the measure on $[-1, 1]$ given by

$$dm_\alpha(t) = (1-t^2)^\alpha\, dt,$$

so that $\{P_n^{(\alpha,\alpha)} : n \geq 0\}$ is a orthogonal basis of $L^2(m_\alpha)$. From (4.3.3) in [23] we know that the normalization constants h_n satisfy

$$h_n \sim A\, n^{-1} \text{ as } n \to \infty \tag{5}$$

3.2. **Uniform Boundedness.** Suppose there is a number $1 < q \leq \infty$ and some positive number A with

$$\|P_n^{(\alpha,\alpha)}\|_{L^q(m_\alpha)} \geq cn^A, \qquad \forall n \geq 1.$$

The formation of the coefficient

$$F \mapsto c_n(F) = \int_{-1}^{1} F(t) P_n^{(\alpha,\alpha)}(t) dm_\alpha(t)$$

is then a bounded linear functional on the dual of $L^q(m_\alpha)$ with norm bounded below by a constant multiple of n^A. The uniform boundedness principle implies that for p conjugate to q and each $0 \leq \varepsilon < A$ there is an $F \in L^p(m_\alpha)$ so that

$$c_n(F)/n^\varepsilon \to \infty \text{ as } n \to \infty. \tag{6}$$

3.3. **Cantor-Lebesgue Theorem.** This idea is explained in [19] and is based on [24, Section IX.1]. Suppose we have a sequence of functions F_n on an interval in the real line with the asymptotic property

$$F_n(\theta) = c_n \left(\cos(M_n\theta + \gamma_n) + \mathbf{o}(1)\right), \qquad \forall n \geq 0$$

uniformly on a set E of finite positive measure, and with $M_n \to \infty$ as $n \to \infty$. Integrating $|F_n|^2$ over E gives

$$\begin{aligned}\int_E |F_n(\theta)|^2\, d\theta &= |c_n|^2 \left(\int_E \cos^2(M_n\theta + \gamma_n)\, d\theta + \mathbf{o}(1)\right) \\ &= |c_n|^2 \left(\frac{|E|}{2} + \frac{e^{2i\gamma_n}}{4}\widehat{\chi_E}(2M_n) + \frac{e^{-2i\gamma_n}}{4}\widehat{\chi_E}(-2M_n) + \mathbf{o}(1)\right).\end{aligned}$$

The Riemann-Lebesgue Theorem [24, Thm. II.4.4] says that the Fourier transforms $\widehat{\chi_E}(\pm 2M_n) \to 0$ as $M_n \to \infty$. If we know that there is some function G for which $|F_n(\theta)| \leq G(n)$ uniformly on E for all n then there is an $n_0 > 0$ for which

$$\frac{|E|}{4}|c_n|^2 \leq \int_E |F_n(\theta)|^2\, d\theta \leq G(n)^2|E|, \quad \forall n \geq n_0.$$

This shows that $|c_n| \leq 2G(n)$ for all $n \geq n_0$.

3.4. **Asymptotics.** Theorem 8.21.8 in Szegő's book[23] gives the following asymptotic behaviour for the Jacobi polynomials $P_n^{(\alpha,\alpha)}$. For $\alpha \geq -1/2$ and $\varepsilon > 0$ the following estimate holds uniformly for all $\varepsilon \leq \theta \leq \pi - \varepsilon$ and $n \geq 1$.

$$P_n^{(\alpha,\alpha)}(\cos\theta) = n^{-1/2} k(\theta) \cos\left(M_n\theta + \gamma\right) + \mathbf{O}\left(n^{-3/2}\right). \tag{7}$$

Here $k(\theta) = \pi^{-1/2}\left(\sin(\theta)/2\right)^{-\alpha-1/2}$, $M_n = n + (2\alpha+1)/2$, and $\gamma = -(\alpha+1/2)\pi/2$.

From Egoroff's theorem and Lemma 2.1 we can say that if the series (4) is Cesàro summable of order δ on a set of positive measure in S^d then there is a set of positive measure $E \subset [0, \pi]$ on which

$$\left| c_n(f_0) h_n^{-1} P_n^{(\alpha,\alpha)} (\cos \theta) \right| \le A n^\delta$$

and hence

$$\left| c_n(f_0) n^{(1/2)-\delta} \left(\cos \left(M_n \theta + \gamma \right) + \mathbf{O}(n^{-1}) \right) \right| \le A \tag{8}$$

uniformly for $\theta \in E$. The argument of subsection 3.3 shows that

$$\left| c_n(f_0) n^{(1/2)-\delta} \right| \le A, \qquad \forall n \ge 1. \tag{9}$$

Lemma 3.1. *If f is a zonal function on the unit sphere whose spherical harmonic expansion is Cesàro summable of order δ on a set of positive measure, then there is a constant $A > 0$ for which*

$$|c_n(f_0)| \le A n^{\delta - (1/2)}, \quad \forall n \ge 1.$$

3.5. **Norm Estimates.** Markett[15] has calculated estimates on the L^p norms of Jacobi polynomials. Let

$$q_c = \frac{4(\alpha+1)}{2\alpha+1} = \frac{2d}{d-1}.$$

Equation (2.2) in [15] gives the following lower bounds on these norms.

Lemma 3.2. *For real number $\alpha > -1/2$, $1 \le q < \infty$, and $r > -1/q$,*

$$\left(\int_0^1 \left| P_n^{(\alpha,\alpha)}(x) \right|^q (1-x)^\alpha \, dx \right)^{1/q} \sim \begin{cases} n^{-1/2} & \text{if } q < q_c, \\ n^{-1/2} (\log n)^{1/q} & \text{if } q = q_c, \\ n^{\alpha - (2\alpha+2)/q} & \text{if } q > q_c. \end{cases}$$

Notice that these integrals are taken over $[0, 1]$ rather than all of $[-1, 1]$.

4. Main Result

Theorem 4.1. *For each $1 \le p < p_c = 2d/(d+1)$,*

$$0 \le \delta < \frac{d}{p} - \frac{d+1}{2},$$

and $y \in S^d$, there is a function in $L^p(S^d)$ which is zonal about y, supported in the hemisphere $\{x : x \cdot y \ge 0\}$, and whose spherical harmonic expansion has Cesàro and Riesz means which diverge almost everywhere.

Proof. Suppose that a series (4) has Cesàro means of order δ which converge on a set of positive measure. Then Lemma 3.1 implies that

$$c_n(f_0) = \mathbf{O}\left(n^{\delta-(1/2)}\right), \quad \text{as } n \to \infty. \tag{10}$$

Compare this inequality with the last line of Lemma 3.2 and section 3.2. If $q > q_c$, $(1/p) + (1/q) = 1$ and

$$\alpha - \frac{(2\alpha + 2)}{q} > \delta - \frac{1}{2}$$

then there must be a zonal function $f \in L^p(S^d)$ with f_0 supported on $[0, 1]$ for which the estimate (10) fails. Remembering the definition of α in terms of the dimension d, we are considering

$$\delta - \frac{1}{2} < \frac{d-1}{2} - d\left(1 - \frac{1}{p}\right)$$

which means

$$\delta < \frac{d}{p} - \frac{(d+1)}{2}.$$

□

Remark 4.1. In [19] we applied this technique to produce an analogous theorem for Laguerre expansions.

5. Central function on $SU(2)$

We conclude with a simple three dimensional example. Suppose that $G = SU(2)$ is equipped with the normalized translation invariant measure μ and that T is the maximal torus of diagonal elements of G. For each $\ell \in \widehat{G} = \{k/2 \,:\, k \in \mathbb{Z}, k \geq 0\}$ there is an irreducible unitary representation of G with dimension $2\ell + 1$ and character

$$\chi_\ell\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = \frac{\sin\left((2\ell+1)\theta\right)}{\sin\left(\theta\right)}.$$

Every central function on G is determined by its restriction to T. The Fourier series of central functions are expansions in the characters. If $f \in L^1(G, \mu)$ is central then

$$f \sim \sum_{\ell=0}^{\infty} c_\ell \chi_\ell \tag{11}$$

with

$$c_\ell = \int_G f(x)\overline{\chi_\ell(x)}\, d\mu(x), \qquad \forall \ell \in \widehat{G}. \tag{12}$$

In [8] and [10], Dooley, Giulini, Soardi, and Travaglini estimated the Lebesgue norms of characters of compact Lie groups. The group $SU(2)$ provides the simplest case of these estimates. For each $q > 3$

$$\|\chi_\ell\|_q \geq c(2\ell + 1)^{1-3/q}, \qquad \forall \ell \in \widehat{G}. \tag{13}$$

If $1/p + 1/q = 1$ then

$$1 - \frac{3}{q} = 1 - 3\left(1 - \frac{1}{p}\right) = \frac{3}{p} - 2.$$

Uniform boundedness then says that if $1 \leq p < 3/2$ and $a < (3/p) - 2$ then there is a central function $f \in L^p(G)$ for which the coefficients in (11) have

$$c_\ell/(2\ell+1)^a \text{ unbounded as } \ell \to \infty.$$

Suppose that (11) is Cesàro summable of order δ on a set of positive measure. Then Lemma 2.1 says that

$$c_\ell \sin\left((2\ell+1)\theta\right) = \mathbf{O}(\ell^\delta) \text{ as } \ell \to \infty,$$

on a set of positive measure. The Cantor-Lebesgue Theorem then says that

$$c_\ell = \mathbf{O}(\ell^\delta) \text{ as } \ell \to \infty.$$

Theorem 5.1. *For $1 \leq p < 3/2$ and $0 \leq \delta < (3/p) - 2$ there is a central function $f \in L^p(SU(2))$ for which the Cesàro and Riesz means of order δ are divergent almost everywhere.*

This shows the sharpness of results in Clerc's paper [6].

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DERIVATION OF MONOGENIC FUNCTIONS AND APPLICATIONS

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ABSTRACT. The paper studies two types of results on inducing monogenic functions in $\mathbf{R}_1^n$. One is based on M^{c}Intosh's formula and the other is along the line of Fueter's Theorem. Applications are summarized and a new application on monogenic sinc function interpolation is introduced.

1. BACKGROUND

Denote by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the *basic elements* that satisfy

$$\mathbf{e}_i^2 = -1, \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i,\ i, j = 1, 2, \ldots, n,\ i < j.$$

We will work on the following spaces:

$$\mathbf{R}^n = \{\underline{x} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n : x_i \in \mathbf{R}, i = 1, \ldots, n\},$$

$$\mathbf{R}_1^n = \{x = x_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^n\},$$

$\mathbf{R}^{(n)}$ is the Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ over the real number field $\mathbf{R}$;

$\mathbf{C}^{(n)}$ is the Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ over the complex number field $\mathbf{C}$.

We adopt the notation $x \in \mathbf{R}^{(n)}$(or $\mathbf{C}^{(n)}$) implies $x = \sum_s x_s\mathbf{e}_s$, where $x_s \in \mathbf{R}$ (or $\mathbf{C}$), and s runs over all the possible ordered sets

$$s = \{0 \le j_1 < \cdots < j_k \le n\},\ \text{or}\ s = \emptyset, \qquad \text{and}$$

$$\mathbf{e}_s = \mathbf{e}_{j1} \cdots \mathbf{e}_{jk}, \quad \mathbf{e}_0 = \mathbf{e}_\emptyset = 1.$$

The functions we will study will be defined in subsets of $\mathbf{R}_1^n$, and take their values in $\mathbf{R}^{(n)}$ or $\mathbf{C}^{(n)}$.

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The Dirac operator D for functions in $\mathbf{R}_1^n$ is defined by

$$D = D_0 + \underline{D},\ D_0 = \frac{\partial}{\partial x_0},\ \underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n.$$

It applies from the left- and right- hand sides to the function, in the manners

$$Df = \sum_{i=0}^{n}\sum_{s}\frac{\partial f_s}{\partial x_i}\mathbf{e}_i\mathbf{e}_s \text{ and } fD = \sum_{i=0}^{n}\sum_{s}\frac{\partial f_s}{\partial x_i}\mathbf{e}_s\mathbf{e}_i,$$

respectively. If $Df = 0$, then f is said to be left-monogenic; and, if $fD = 0$, then right-monogenic. If f is both left- and right- monogenic, then it is said to be monogenic.

Examples:

(1) The case $n = 1$ corresponds to the complex number field: $\mathbf{e}_1 = \mathbf{i}$, $D = \frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}$, $f(z) = u(x,y) + \mathbf{i}v(x,y)$ and $Df = 0$ if and only if the Cauchy-Riemann equations hold:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}. \end{cases}$$

(2) The case $n = 2$ corresponds to the space of Hamilton quaternions:

$$\vec{\mathbf{i}} = \mathbf{e}_1, \vec{\mathbf{j}} = \mathbf{e}_2, \vec{\mathbf{k}} = \mathbf{e}_1\mathbf{e}_2,$$

$$q = q_0 + q_1\vec{\mathbf{i}} + q_2\vec{\mathbf{j}} + q_3\vec{\mathbf{k}}, q_k \in \mathbf{R}.$$

Profound studies on Clifford analysis have been conducted since Fueter's school in the 1930's till the present time (see, for instance, [Ma], [CS] and [Q1] and their references).

(3) Let $u_j(x), j = 0, 1, \ldots, n$, be defined in $\mathbf{R}_1^n$ with values in $\mathbf{C}$. Set

$$U = -u_0 + u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$$

Then

$$DU = 0$$

if and only if these functions form a conjugate harmonic system (or satisfy the generalized Cauchy-Riemann equations, see [St] and [KQ1]):

$$\begin{cases} \displaystyle\sum_{j=0}^{n}\frac{\partial u_j}{\partial x_j} = 0 \\ \dfrac{\partial u_k}{\partial x_j} = \dfrac{\partial u_j}{\partial x_k}, 0 \le k < j \le n. \end{cases}$$

For monogenic functions there hold Cauchy's Theorem and Cauchy's formula. The Cauchy kernel in the context is $\mathbf{E}(x) = \frac{\overline{x}}{|x|^{n+1}}$, where for $x = x_0 + \underline{x}$, we denote $\overline{x} = x_0 - \underline{x}$ .. It is observed that the Clifford structure of $\mathbf{R}_1^n$ is the "true" analogue of the one complex variable structure of $\mathbf{R}_1^1$.

2. C-K Extension and M^{c}Intosh's Formula

It can be proved that if we have a real analytic function defined in an open set O of $\mathbf{R}^n$, then we can always monogenically extend it to an open set Q of $\mathbf{R}_1^n$ where $O = \mathbf{R}^n \cap Q$ (*C-K extension*, see, for instance [BDS]). The extension can be realized by the operation $\mathbf{e}^{-x_0\underline{D}} f(\underline{x})$, understood in the symbolic way. In fact, formally we have

$$\begin{aligned} D(e^{-x_0\underline{D}} f(\underline{x})) &= (D_0 + \underline{D})(e^{-x_0\underline{D}} f(\underline{x})) \\ &= (-\underline{D})e^{-x_0\underline{D}} f(\underline{x}) + \underline{D}e^{-x_0\underline{D}} f(\underline{x}) = 0. \end{aligned}$$

Examples:

(1) If $f(x) = x_j$, then

$$e^{-x_0\underline{D}}(x_j) = (1 + (-x_0\underline{D}) + \frac{1}{2!}(-x_0\underline{D})_2 + \cdots)x_j = x_j\mathbf{e}_0 - x_0\mathbf{e}_j \overset{\triangle}{=} z_j.$$

(2) The extension of $x_i x_j, i \neq j$, is

$$\frac{1}{2}(z_i z_j + z_j z_i),$$

etc.

In practice the C-K extension and the related forms are, in general, complicated and not easy to use. On the contrary, M^{c}Intosh's formula, somehow plays the role of Fourier-Laplace transform in $\mathbf{R}_1^n$, has been playing a crucial role in a number of questions in function theory ([LMcQ], [PQ], [Q4], [KQ1], [KQ2]). The formula first appeared in late 1980's ([Mc1]) and formally published in [Mc2] and [LMcQ] in 1994. The formula involves a set of notations: If f is defined in $\mathbf{R}^n$ with Fourier transform, then the possible monogenic extension of f is given by

$$f(x) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^n} e(x,\underline{\xi}) \hat{f}(\underline{\xi})d\underline{\xi}, \qquad \text{(M}^c\text{Intosh's formula)},$$

provided that the integral on the right-hand-side is properly defined, where

$$e(x,\underline{\xi}) = e^{\mathbf{i}\underline{x}\cdot\underline{\xi}}\{e^{-x_0|\underline{\xi}|}\chi_+(\underline{\xi}) + e^{x_0|\underline{\xi}|}\chi_-(\underline{\xi})\},$$

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}(1 + \mathbf{i}\frac{\underline{\xi}}{|\underline{\xi}|}), \quad x = x_0 + \underline{x}, \quad \underline{x}, \underline{\xi} \in \mathbf{R}^n.$$

In [BDS] a wide range of similar notions are introduced. It is exactly M^cIntosh's form, however, that has been effectively used, especially in problems related to Fourier transformation.

In the formulas for the projections $\chi_{\pm}$, if we take $n = 1$ and $\mathbf{e}_1 = -\mathbf{i}$, then we have

$$\chi_{\pm}(\underline{\xi}) = \pm \mathrm{sgn}\xi,$$

where $\xi = \mathbf{i}\underline{\xi}$.

This indicates that the formula provides a decomposition of a function into functions similar to those in the Hardy spaces. Indeed, we have,

$$f(x) = f^{+}(x) + f^{-}(x),$$

where

$$f^{\pm}(x) \stackrel{\triangle}{=} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \mathbf{e}^{\pm}(x,\underline{\xi}) \stackrel{\wedge}{f} (\underline{\xi}) d\underline{\xi},$$

$$\mathbf{e}^{\pm}(x,\underline{\xi}) = \mathbf{e}^{i\underline{x}\cdot\underline{\xi}} \mathbf{e}^{\mp x_0 |\underline{\xi}|} \chi_{\pm}(\underline{\xi}).$$

We can further show that for $x_0 > 0$,

$$f^{+}(x) = \frac{1}{w_n} \int_{\mathbf{R}^n} E(x - \underline{y}) \cdot f(\underline{y}) d\underline{y};$$

while $f^{-}(x), x_0 > 0$, is the monogenic extension of $f^{-}(x)$ for $x_0 < 0$, where the latter is also of the Cauchy's integral form of f. For $x_0 < 0$ we have the analogous notation.

Under the context of the classical Paley-Wiener Theorem in the case $n = 1$, viz. $f \in L^2(\mathbf{R})$ and

$$\mathrm{supp} \stackrel{\wedge}{f} (\xi) \subset [-\delta, \delta], \quad \delta > 0,$$

there follows

$$f(z) = f^{+}(z) + f^{-}(z).$$

For $z = x + iy,\ y > 0$, we have

$$e^{-y|\underline{\xi}|}\chi_{[0,\delta]}(\underline{\xi}) = e^{-y\xi}\chi_{[0,\delta]}(\underline{\xi}), \qquad e^{y|\underline{\xi}|}\chi_{[-\delta,0]}(\underline{\xi}) = e^{-y\xi}\chi_{[-\delta,0]}(\underline{\xi}),$$

and so

$$f^{+}(z) = \frac{1}{2\pi} \int_0^{\delta} \mathbf{e}^{ix\xi} \mathbf{e}^{-y\xi} \stackrel{\wedge}{f} (\xi) d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt,$$

and

$$f^{-}(z) = \frac{1}{2\pi} \int_{-\delta}^{0} \mathbf{e}^{ix\xi} \mathbf{e}^{-y\xi} \stackrel{\wedge}{f} (\xi) d\xi,$$

where $f^{-}(z)$ is well defined, but not be expressible by a Cauchy integral. In fact, since $y = \mathrm{Im}\ z > 0$, $f^{-}(z)$ is the holomorphic extension to the upper-half complex plane of the Cauchy integral of f in the lower-half complex plane. By virtue of M^{c}Intosh's formula we have exactly the same notion in $\mathbf{R}_1^n$.

We will mention two applications of M^{c}Intosh's formula.

(1) Paley-Wiener Theorem in $\mathbf{R}_1^n$

In a recent paper we proved the following theorem ([KQ1]).

Theorem. *Let $f \in L^2(\mathbf{R}^n)$. Then f can be monogenically extended to $\mathbf{R}_1^n$ with the estimate*

$$|f(x)| \leq c\mathbf{e}^{\mathbf{R}|x|}$$

if and only if

$$\text{supp}\ \stackrel{\wedge}{f} \subset \overline{B(0,R)},$$

where

$$B(0,R) = \{x \in \mathbf{R}^n : |\underline{x}| < R\}.$$

In the case we have

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e(x,\underline{\xi})\hat{f}(\underline{\xi})d\underline{\xi}, \quad x \in \mathbf{R}_1^n.$$

In the literature higher dimensional versions of the Paley-Wiener theorem have been sought (see the references of [KQ1]). We wish to make the point that the version with the Clifford algebra setting provides the precise analogue. The commonly adopted proofs of the classical Paley-Wiener Theorem are not readily applicable to the Clifford setting owing to the defect that products of monogenic functions are no longer again monogenic in general. However, a particular proof for the one complex variable case can be closely followed through a non-trivial computation based on M^{c}Intosh's formula([KQ1]).

(2) Monogenic sinc function with Shannon sampling for functions in the Paley-Wiener classes

Define the class of functions

$$PW(R) = \Big\{ f : \mathbf{R}_1^n \to \mathbf{C}^{(n)} : \quad f \text{ is monogenic in the whole } \mathbf{R}_1^n \text{ and satisfies } |f(x)| \leq C\mathbf{e}^{R|x|} \Big\}.$$

The monogenic sinc function is defined to be

$$\text{sinc}(x) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} \mathbf{e}(x,\underline{\xi})d\underline{\xi}.$$

The following exact interpolation of functions in the $PW(R)$ classes is proved in ([KQ2]).

Theorem. *If $f \in PW(\frac{\pi}{h})$, then*

$$f(x) = \sum_{\underline{k}\in\mathbf{Z}^n} f(h\underline{k})\,\text{sinc}\left(\frac{x - h\underline{k}}{h}\right),$$

where the convergence is in the pointwise sense independent of the order of summation.

The proof is based on estimates of the monogenic sinc function derived from M^c^Intosh's formula.

3. Fueter's Theorem and Generalizations

This addresses the problem of deriving monogenic and harmonic functions from those of the same kind but in lower dimensional spaces.

Let $f^0(z)$ be a function of one complex variable analytic in an open set O of the upper-half complex plane $\mathbf{C}^+$. If $f(z) = u(x, y) + \mathbf{i}v(x, y)$, $z = x + iy$, we introduce

$$\vec{f}^0(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|).$$

and set

$$\tau(f^0)(x) = \Delta^{\frac{n-1}{2}} \vec{f}^0(x), x \in \mathbf{R}_1^n.$$

Theorem of Fueter (1935). *When $n = 3$, interpreted as the quaternionic space, the mapping τ maps an analytic function $f^0(z)$ in O to a quaternionic monogenic function in*

$$\vec{O} = \{q = q_0 + \underline{q} : q_0 + \mathbf{i}|\underline{q}| \in O\}.$$

Theorem of Sce (1957). *For n being an odd integer the mapping τ maps $f^0(z)$ to a monogenic function in*

$$\vec{O} = \{x = x_0 + \underline{x} : x_0 + \mathbf{i}|\underline{x}| \in O\}.$$

These results were extended in [Q2] in 1997 to the cases n being an integer and the operator $\Delta^{\frac{n-1}{2}}$ interpreted as the Fourier multiplier operator with symbol $|\xi|_{n-1}$. We note that

$$\tau(\frac{1}{z})(x) = E(x) = \frac{1}{|x|^{n+1}}.$$

In [Q2-3] for any integer $n \geq 2$ a corresponding relationship between the functions $f^0(z) = z^k$ and certain monogenic functions $P^{(k)}(x)$ of homogeneity of degree k is established:

$$\tau(\frac{1}{z^k})(x) = P^{(-k)}(x), k = 1, 2, \ldots,$$

and

$$P^{(k-1)} = I(P^{(-k)}), k = 1, 2, \ldots,$$

where I is the Kelvin inversion defined by

$$If(x) = E(x)f(x^{-1}).$$

It is noted that if n is an odd integer, then

$$P^{(k-1)} = \tau(z^{n+k-2}).$$

The sequence $P^{(k)}, k \in \mathbf{Z}$, is used to establish the bounded holomorphic functional calculus of the Dirac operator on Lipschitz perturbations,denoted by D_Σ, of the unit sphere in $\mathbf{R}_1^n$ (and similarly on $\mathbf{R}^n$).

We now describe the result. Set

$$\mathbf{S}_w = \{0 \neq z \in \mathbf{C} : z = x + iy, \frac{|y|}{|x|} < \tan w\},\ 0 < w < \frac{\pi}{2},$$

$$\tan w > \text{the Lipschitz constant of } \Sigma,$$

$$\mathbf{H}^\infty(\mathbf{S}_w) = \{b : \mathbf{S}_w \to \mathbf{C} : f^o \text{ is bounded and analytic in } \mathbf{S}_w\}.$$

Given $b \in \mathbf{H}^\infty(\mathbf{S}_w)$, and set, formally,

$$b(D_\Sigma)f = \frac{1}{2\pi i}\int_\gamma b(\xi)(I - D_\Sigma)^{-1} d\xi f,$$

where γ is a certain curve in $\mathbf{S}_w$ surrounding the spectrum of D_Σ. The operators $b(D_\Sigma)$ are proved to be equal to the Fourier multiplier operators

$$M_b f(x) = \sum_{k=1}^\infty b(k) P_k f(x) + \sum_{k=1}^\infty b(-k) Q_k f(x),$$

where $P_k f$ and $Q_k f$ are projections of f onto the spaces of monogenic functions of homogeneity degree k and $-k$, respectively. They are also equal to the singular integral operators

$$S_\Phi f(x) = \lim_{n\to\infty}\left\{\frac{1}{w_n}\int_{|x-y|>\epsilon} \Phi(y^{-1}x)E(y)n(y)f(y)d\sigma(y) + \Phi^1(\epsilon, x)f(x)\right\},$$

where in a certain sense $\Phi = \overset{\vee}{b}$ (the inverse Fourier transform of b) and $\Phi^1(\epsilon, x) =$ the average of Φ on the sphere centered at x of radius ϵ.

That is

$$b(D_{\sum}) = M_b = S_\phi.$$

The boundedness of the operators $b(D_\Sigma)$, $b \in \mathbf{H}^\infty(S_w)$, is proved through their singular integral expressions S_Φ based on the estimates of the kernels Φ and Φ^1. The derivation of the estimates are reduced to the similar estimates in the one complex variable case via the correspondence between the functions z^k and $P^{(k)}$ ([Q5]).

On Lipschitz perturbations of higher dimensional spheres the theory cannot be done through the Poisson Summation method based on the graph case as in the unit circle case ([Q5]). It encountered some difficulties and hence was first achieved in the quaternionic space ([Q1]), and then in general Euclidean spaces ([Q3]).

Further generalizations of Fueter's Theorem include the following.

(i) In a recent paper F.Sommen proved that if n is an odd positive integer and $x \in \mathbf{R}_1^n$, then for $f^0(z) = u(s,t) + \mathbf{i}v(s,t), z = s + it$, analytic in an open set $O \subset \mathbf{C}^+$, then for $x \in \vec{O}$

$$D\Delta^{k+\frac{n-1}{2}}((u(x_0, |x|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|))P_k(\underline{x})) = 0,$$

where P_k is any polynomial in $\underline{x}$ of homogeneity k, left-monogenic with respect to the Dirac operator $\underline{D}$ ([So]).

(ii) K.I.Kou and T.Qian extended Sommen's result to the cases when n is an even positive integer and Sommen extended his result to the cases $k + \frac{n-1}{2}$ being non-negative integers, no matter whether k is an integer ([KQS]).

(iii) The derivation of monogenic functions can be reduced to that of harmonic functions, based on the following observations.

A. If h is harmonic in $x_0, x_1, \ldots, x_n$, then $\overline{D}h$ is monogenic, where $\overline{D} = D_0 - \underline{D}$.

B. If f is monogenic, then there exists a harmonic function h such that $f = \overline{D}h$.

The following result for harmonic functions is obtained in a recent paper of T.Qian and F.Sommen ([QS]).

Denote

$$\underline{x}^{(r)} = x_1^{(r)}\mathbf{e}_1^{(r)} + \cdots + x_{p_r}^{(r)}\mathbf{e}_{p_r}^{(r)} \in \mathbf{R}^{p_r},$$

where $r = 1, \ldots, d, \sum_{r=1}^d p_r = m$, and

$$\mathbf{e}_i^{(r)}\mathbf{e}_{i'}^{(r')} = -\mathbf{e}_{i'}^{(r')}\mathbf{e}_i^{(r)}, \text{wherever } (r,i) \neq (r', i').$$

Let $h(s_1, \ldots, s_d)$ be a harmonic function in the d variables $s_1, \ldots, s_d$. Them, if $p_r, r = 1, \ldots, d$, are odd and $m = \sum_{r=1}^d p_r$ is even, then

$$\Delta^{\frac{m}{2}} h(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|) = 0,$$

where Δ is the Laplacian for all the m variables $x_i^r, r = 1, \ldots, d$, $i = 1, \ldots, p_r$.

(iv) The latest result along this line is by K.I.Kou and T.Qian ([KQ3]), as follows.

In the above notation we have

$$\Delta^{(k_1+\ldots+k_d)+\frac{m}{2}}[h(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|)P_{k_1}^{(1)} \cdots P_{k_d}^{(d)}(\underline{x}^{(d)})] = 0,$$

where for any $r = 1, 2, \ldots, d$, $P_{k_r}^{(r)}(\underline{x}^{(r)})$ is a left-monogenic functions with respect to $\underline{D}^{(r)}$, homogeneous of degree k_r, where k_r is any non-negative integer.

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ASYMPTOTICS OF SEMIGROUP KERNELS

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Dedicated to Alan McIntosh on occasion of his 60-th birthday

ABSTRACT. We review the large time behaviour of the semigroup kernel associated with a homogeneous operator on a Lie group with polynomial growth. We consider complex second-order operators and two classes of higher order operators.

1. INTRODUCTION

There is a vast literature on second-order elliptic operators on Lie groups with real symmetric coefficients. (See [Rob], [VSC], and references cited therein.) The closure of such an operator generates a semigroup which is holomorphic in a sector, the semigroup has a smooth kernel and the kernel, together with all its derivatives, satisfies the canonical Gaussian upper bounds for small time. These results have been extended to various other classes of complex subelliptic operators of any order on a Lie group and in particular one has again the Gaussian upper bounds for small time. If the operator is a real symmetric pure second-order operator and the Lie group has polynomial growth then the kernel satisfies the Gaussian upper bounds for all time. The aim of this note is to indicate the difficulties that one can expect for the large time canonical Gaussian upper bounds associated to other classes of operators on Lie groups with polynomial growth. In particular we discuss the class of pure second-order operators with complex constant coefficients and two classes of higher order homogeneous operators.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $a_1, \ldots, a_{d'}$ be an algebraic basis for $\mathfrak{g}$, i.e., independent elements which together with their multi-commutators up to order s span $\mathfrak{g}$. The smallest number s for which this is valid is called the **rank** of the algebraic basis. Let dg be a (left) Haar measure on G. For all $p \in [1, \infty]$ let L denote the left regular representation in $L_p(G) = L_p(G\,; dg)$. For all $i \in \{1, \ldots, d'\}$ let A_i be the infinitesimal generator of the one parameter group $t \mapsto L(\exp(-t a_i))$. We also need multi-index notation since the A_i do not commute in general. Set $J(d') = \bigcup_{n=0}^{\infty} \{1, \ldots, d'\}^n$ and

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if $\alpha = (i_1, \ldots, i_n) \in J(d')$ set $A^\alpha = A_{i_1} \ldots A_{i_n}$ and $|\alpha| = n$. Associated to the algebraic basis there is a modulus on G, i.e., the distance to the identity element e of G. Introduce D as the set of functions

$$W = \{\psi \in C_c^\infty(G) : \psi \text{ is real and } \sup_{g \in G} \sum_{k=1}^{d'} |(A_k\psi)(g)|^2 \leq 1\} \quad .$$

For all $g \in G$ define

$$|g|' = \sup_{\psi \in W} |\psi(g) - \psi(e)| \quad .$$

Finally, for all $\rho > 0$ let the **volume** $V'(\rho)$ be the Haar measure of the ball $\{g \in G : |g|' < \rho\}$.

The Lie algebra $\mathfrak{g}$ is called **nilpotent** if there exists an $n \in \mathbf{N}$ such that

$$[b_1, [b_2, \ldots, [b_{n-1}, b_n] \ldots]] = 0$$

for all $b_1, \ldots, b_n \in \mathfrak{g}$. If $\mathfrak{g}$ is nilpotent then define the **rank** r of $\mathfrak{g}$ by

$$r = \max\{n \in \mathbf{N} : \exists_{b_1, \ldots, b_n \in \mathfrak{g}} [b_1, [b_2, \ldots, [b_{n-1}, b_n] \ldots]] \neq 0\} \quad .$$

Let $\tilde{\mathfrak{g}}(d', r)$ be the nilpotent Lie algebra with maximal dimension, d' generators and rank r. We denote the generators by $\tilde{a}_1, \ldots, \tilde{a}_{d'}$ and the associated infinitesimal generators by $\widetilde{A}_1, \ldots, \widetilde{A}_{d'}$. Let $\widetilde{G}(d', r)$ be the connected simply connected Lie group with Lie algebra $\tilde{\mathfrak{g}}(d', r)$.

Now we are able to introduce the operators we want to consider. Let $m \in 2\mathbf{N}$. On a general Lie group G the operator

$$H = \sum_{|\alpha| \leq m} c_\alpha A^\alpha$$

with $c_\alpha \in \mathbf{C}$ and domain $D(H) = \bigcap_{|\alpha| \leq m} D(A^\alpha)$ is called **subcoercive of step** r if the comparable operator

$$\widetilde{H} = \sum_{|\alpha| = m} c_\alpha \widetilde{A}^\alpha$$

on $L_2(\widetilde{G}(d', r))$ satisfies a Gårding inequality, i.e.,

$$\mathrm{Re}(\tilde{\varphi}, \widetilde{H}\tilde{\varphi}) \geq \tilde{\mu} \sum_{|\alpha| = m/2} \|\widetilde{A}^\alpha \tilde{\varphi}\|_2^2$$

for all $\tilde{\varphi} \in C_c^\infty(\widetilde{G}(d', r))$, for some $\tilde{\mu} > 0$. Note that the condition is on the principal part of the operator and is independent of the lower order terms. If H is subcoercive of step r then H is also subcoercive of step $r - 1$ (see [ElR1], Corollary 3.6).

Example 1.1. Let $c_{ij} \in \mathbf{C}$ and suppose there exists a $\mu > 0$ such that

$$\mathrm{Re} \sum_{i,j=1}^{d'} c_{ij} \, \overline{\xi_i} \, \xi_j \geq \mu \, |\xi|^2 \tag{1}$$

for all $\xi \in \mathbf{C}^{d'}$. Then the operator $-\sum_{i,j=1}^{d'} c_{ij} \, A_i \, A_j$ is a subcoercive operator of step r for all $r \in \mathbf{N}$. This is easy to verify. Conversely, if $-\sum_{i,j=1}^{d'} c_{ij} \, A_i \, A_j$ is a subcoercive operator of step r for some $r \geq 2$ then (1) is valid. (See [ElR1], Proposition 3.7.)

For all $m \in 2\mathbf{N}$ the operators $\left(-\sum_{i=1}^{d'} A_i 2\right)^{m/2}$ and $(-1)^{m/2} \sum_{i=1}^{d'} A_i^m$ are also subcoercive operators of step r for all $r \in \mathbf{N}$, but the proof is not trivial. (See [ElR2], Example 4.4.)

The group $\widetilde{G}(2,2)$ is the Heisenberg group and the operator $-\widetilde{A}_1 2 - \widetilde{A}_2 2 - 10 i [\widetilde{A}_1, \widetilde{A}_2]$ is subcoercive of step 1, but not subcoercive of order r for any $r \geq 2$.

The small time theory is well developed. Set

$$G_{b,t}^{(m)}(g) = V'(t)^{-1/m} \, e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $b > 0$, $t > 0$ and $g \in G$.

Theorem 1.2. *Let $a_1, \ldots, a_{d'}$ be an algebraic basis of rank r for the Lie algebra of a Lie group G. Let $m \in 2\mathbf{N}$ and H an m-th order subcoercive operator of step s with $s \geq r$. Then one has the following.*

I. *The closure of H generates a semigroup S on L_p for all $p \in [1, \infty]$.*

II. *The semigroup S is holomorphic.*

III. *The semigroup has a smooth rapidly decreasing p-independent kernel K, i.e., $S_t \varphi = K_t * \varphi$ for all $\varphi \in L_p$ and $t > 0$.*

IV. *For all $\alpha \in J(d')$ there exist $b, c > 0$ and $\omega \geq 0$ such that*

$$|A^\alpha K_t| \leq c \, t^{-|\alpha|/m} \, e^{\omega t} \, G_{b,t}^{(m)} \tag{2}$$

for all $t > 0$.

V. *If $p \in \langle 1, \infty \rangle$ then H is closed on L_p. Moreover, if the semigroup S is uniformly bounded on L_p then for all $N \in \mathbf{N}$ one has $D(H^{N/m}) = \bigcap_{|\alpha|=N} D(A^\alpha)$.*

Proof. See [ElR1], Theorems 2.5 and 4.1, and for Statement V see [BER]. □

The bounds (2) are optimal for $t \in \langle 0, 1]$ and describe the Gaussian decay. For large t the factor $e^{\omega t}$ reflects the semigroup property and the contribution of the lower order terms in H. Each derivative of K

gives a $t^{-1/m}$-singularity. We say that $A^\alpha K$ has the **canonical** (large time) Gaussian upper bounds if the bounds (2) are valid for all $t > 0$ with $\omega = 0$. On a subclass of Lie groups these bounds are valid for particular operators.

2. Second-order operators

A Lie group is said to have **polynomial growth** if there exist $c > 0$ and $D \in \mathbf{N}_0$ such that the volume $V'(\rho) \leq c\,\rho^D$ for all $\rho \geq 1$ ([Gui] and [Jen]). It turns out that this definition is independent of the choice of the algebraic basis.

Theorem 2.1. *If G has polynomial growth and H is a pure second-order real symmetric subcoercive operator of step* 2 *then there exist $b, c > 0$ such that*

$$|K_t| \leq c\, G^{(2)}_{b,t} \quad \textit{and} \quad |A_i K_t| \leq c\, t^{-1/2}\, G^{(2)}_{b,t}$$

for all $t > 0$ and $i \in \{1, \ldots, d'\}$.

Proof. See [SC]. □

In the situation of Theorem 2.1 the reality of the coefficients implies that the kernel K is positive and by the Beurling-Deny criterium the semigroup S is a contraction semigroup on L_∞. Therefore K is integrable and $\int K_t = 1$ for all $t > 0$. By duality the semigroup is also a contraction semigroup on L_1 and then a Nash inequality provides L_∞-bounds on K_t. Finally by a Davies perturbation the L_∞-bounds for K can be strengthened to the Gaussian bounds of Theorem 2.1.

If the coefficients are complex then K is not positive and there is no version of the Beurling-Deny criterium for complex operators [ABBO]. It is a recent result that the reality of the coefficients in Theorem 2.1 is superfluous.

Theorem 2.2. *If G has polynomial growth and H is a pure second-order* (*complex*) *subcoercive operator of step* 2 *then there exist $b, c > 0$ such that*

$$|K_t| \leq c\, G^{(2)}_{b,t} \quad \textit{and} \quad |A_i K_t| \leq c\, t^{-1/2}\, G^{(2)}_{b,t}$$

for all $t > 0$ and $i \in \{1, \ldots, d'\}$.

Proof. See [DER2]. □

Although the first-order derivatives of the kernel satisfy the canonical large time Gaussian upper bounds, in general higher order derivatives fail to have the canonical large time Gaussian upper bounds.

Theorem 2.3. *Suppose G has polynomial growth and let H be a pure second-order (complex) subcoercive operator of step 2. The following are equivalent.*

I. *There exist $b, c > 0$ such that*
$$|A_i A_j K_t| \leq c\, t^{-1}\, G_{b,t}^{(2)}$$
for all $t > 0$ and $i, j \in \{1, \ldots, d'\}$.

II. *There exists a $c > 0$ such that $\|A_i A_j S_t\|_{2\to 2} \leq c\, t^{-1}$ for all $t > 0$ and $i, j \in \{1, \ldots, d'\}$.*

III. *For all $p \in \langle 1, \infty \rangle$ there exists a $c > 0$ such that $\|A_i A_j \varphi\|_p \leq c\, \|H\varphi\|_p$ for all $\varphi \in D(H)$ and $i, j \in \{1, \ldots, d'\}$.*

IV. *There exist $\sigma \in \langle 0, 1 \rangle$, $c > 0$ and for all $R \in \langle 0, \infty \rangle$ a function $\eta_R \in C^\infty(G)$ such that $0 \leq \eta_R \leq 1$, $\eta_R(g) = 1$ for all $g \in G$ with $|g|' \leq \sigma R$, $\eta_R(g) = 0$ for all $g \in G$ with $|g|' \geq R$ and $\|A^\alpha \eta_R\|_\infty \leq c\, R^{-|\alpha|}$ for all $\alpha \in J(d')$ with $|\alpha| = 2$.*

V. *The Lie algebra of G is the direct product of the Lie algebra of a compact group and a nilpotent Lie algebra.*

Proof. For real symmetric operators this theorem has been proved in [ERS2]. The general case is again in [DER2]. □

In [Ale] Alexopoulos gave an example of a sublaplacian H on the covering group of the Euclidean motion group for which Condition III of Theorem 2.3 fails.

A similar theorem is valid for n derivatives instead of only two derivatives, with $n \geq 3$. In particular, on a nilpotent Lie group all higher order derivatives of the kernel associated to a complex second-order operator of step r, with $r \geq 2$, satisfy the canonical Gaussian upper bounds for large time. That is a special case of the following theorem.

Theorem 2.4. *Let G be a nilpotent Lie group and let r be the rank of its Lie algebra. Let $m \in 2\mathbf{N}$ and let H be a pure m-th order operator which is subcoercive of step r. Then for all $\alpha \in J(d')$ there exist $b, c > 0$ such that*
$$|A^\alpha K_t| \leq c\, t^{-|\alpha|/m}\, G_{b,t}^{(m)}$$
for all $t > 0$.

Moreover, for all $N \in \mathbf{N}$ and $p \in \langle 1, \infty \rangle$ there exists a $c > 0$ such that

$$c^{-1} \max_{|\alpha|=N} \|A^\alpha \varphi\|_p \leq \|H^{N/m} \varphi\|_p \leq c \max_{|\alpha|=N} \|A^\alpha \varphi\|_p \tag{3}$$

for all $\varphi \in D(H^{N/m}) = \bigcap_{|\alpha| \leq N} D(A^\alpha)$.

Proof. See [NRS] or [ERS1]. □

3. Sums of subcoercive operators

Before we describe a more general theorem we first consider an example on $\mathbf{R}^d$.

Example 3.1. Let $\Delta = -\sum_{i=1}^d \partial_i 2$ be the Laplacian on $\mathbf{R}^d$ and $n, m \in 2\mathbf{N}$ such that $n \leq m$. Set

$$H = \Delta^{n/2} + \Delta^{m/2} \quad .$$

Let K, $K^{(n)}$ and $K^{(m)}$ denote the kernels of the semigroups generated by H, $\Delta^{n/2}$ and $\Delta^{m/2}$. Then

$$K_t * \varphi = S_t\varphi = e^{-tH}\varphi = e^{-t\Delta^{n/2}} e^{-t\Delta^{m/2}}\varphi = K_t^{(n)} * K_t^{(m)} * \varphi$$

for all $\varphi \in L_2(\mathbf{R}^d)$ since $\Delta^{n/2}$ and $\Delta^{m/2}$ commute. So $K_t = K_t^{(n)} * K_t^{(m)}$ for all $t > 0$. Hence there exist $b, c > 0$ such that

$$|K_t| \leq c\,(G_{b,t}^{(n)} * G_{b,t}^{(m)})$$

for all $t > 0$ and $x \in \mathbf{R}^d$. These bounds can be reexpressed as follows. Set

$$E_{b,t}^{(m,n)}(x) = (t^{-d/n} \wedge t^{-d/m})(e^{-b(|x|^n t^{-1})^{n/(n-1)}} \vee e^{-b(|x|^m t^{-1})^{m/(m-1)}}) \quad .$$

Then for all $b > 0$ there exist $b', c > 0$ such that

$$G_{b,t}^{(n)} * G_{b,t}^{(m)} \leq c\,E_{b',t}^{(m,n)}$$

and

$$E_{b,t}^{(m,n)} \leq c\,(G_{b',t}^{(n)} * G_{b',t}^{(m)})$$

for all $t > 0$. Thus there are $b, c > 0$ such that $|K_t| \leq c\,E_{b,t}^{(m,n)}$ for all $t > 0$.

Using Fourier analysis it is not hard to show that there exists a $c > 0$ such that

$$c^{-1}\,t^{-\nu}t^{-d/n} \leq \|K_t - K_t^{(n)}\|_\infty \leq c\,t^{-\nu}t^{-d/n}$$

uniformly for all $t \geq 1$, where $\nu = (m-n)/n$. So for large t the kernel $K^{(n)}$ is a first approximation of K. One might hope that one has bounds $|K_t| \leq c\,G_{b,t}^{(n)}$ for suitable $b, c > 0$, uniformly for all $t \geq 1$, but these are not valid by the following argument. If $y \in \mathbf{R}^d$ then the Lebesgue dominated convergence theorem implies that

$$\lim_{t\to\infty} t^{d/m} K_t(t^{1/m}y) = (2\pi)^{-d} \int dp\, e^{-ip\cdot y} e^{-|p|^m}$$

and the integral is not zero for all $y \in \mathbf{R}^d$. But

$$\lim_{t\to\infty} t^{d/m} G_{b,t}^{(n)}(t^{1/m}y) = 0$$

for all $b > 0$ and $y \in \mathbf{R}^d$. Therefore there are no $b, c > 0$ such that $|K_t| \leq c\, G^{(n)}_{b,t}$ uniformly for all $t \geq 1$.

The general case on a nilpotent Lie group is as follows.

Theorem 3.2. *Let G be a nilpotent Lie group and let r be the rank of its Lie algebra. Let $k \in \mathbf{N}\backslash\{1\}$ and $m_1, \ldots, m_k \in 2\mathbf{N}$ with $m_1 > m_2 > \ldots > m_k$. For all $j \in \{1, \ldots, k\}$ let H_{m_j} be a pure m_j-th order subcoercive operator of step r. Set $H = \sum_{j=1}^{k} H_{m_j}$ and let K be the kernel of the semigroup generated by H. Then one has the following.*

I. *For all α there are $b, c > 0$ such that*

$$|A^\alpha K_t| \leq c\,(t^{-|\alpha|/m} \wedge t^{-|\alpha|/\underline{m}})\,(G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})$$

for all $t > 0$ where $m = m_1$ and $\underline{m} = m_k$.

II. *For all $\alpha \in J(d')$ and $p \in \langle 1, \infty\rangle$ there exists a $c > 0$ such that*

$$\|A^\alpha \varphi\|_p \leq c\|H^{|\alpha|/m_j}\varphi\|_p$$

for all $j \in \{1, \ldots, k\}$ and $\varphi \in D(H^{|\alpha|/m_j})$.

III. *For all $\alpha \in J(d')$ there exist $b, c > 0$ such that*

$$|A^\alpha K_t - A^\alpha K^{(\underline{m})}_t| \leq c\,t^{-\nu} t^{-|\alpha|/\underline{m}}\ (G^{(m)}_{b,t} * G^{(\underline{m})}_{b,t})$$

for all $t \geq 1$, where $K^{(\underline{m})}$ denotes the kernel of $H_{\underline{m}}$ and $\nu = (m_{k-1} - m_k)/m_k$.

Proof. See [DER1], Theorems 2.1 and 2.12. □

Again Statement III of Theorem 3.2 indicates that $K^{(\underline{m})}$ is the first order approximation of the kernel K for large t. Thus the large time behaviour is determined by the lowest order terms in H. The kernel K can be bounded by a Gaussian only in a very special case.

Proposition 3.3. *Let $n \in \mathbf{N}\backslash\{1\}$ and adopt the notation of Theorem 3.2. The following are equivalent.*

I. *There exist $b, c > 0$ such that $|K_t| \leq c\, G^{(n)}_{b,t}$ for all $t > 0$.*

II. *$n = m = \underline{m}$ or G is compact and $n \geq m$.*

Proof. See [DER1], Proposition 2.15. □

4. Higher order operators

It follows from Theorem 2.4 and Example 1.1 that for all $m \in 2\mathbf{N}$ the kernel of the semigroup generated by the operator $\Big(-\sum_{i=1}^{d'} A_i 2\Big)^{m/2}$ satisfies canonical Gaussian upper bounds for all time if G is nilpotent. The condition that G is nilpotent can be relaxed.

Theorem 4.1. *Let $a_1, \ldots, a_{d'}$ be an algebraic basis of rank r for the Lie algebra of a Lie group G with polynomial growth. Let $m \in 2\mathbf{N}$ and let K be the kernel of the semigroup generated by the operator $\Big(- \sum_{i=1}^{d'} A_i 2\Big)^{m/2}$. Then there exist $b, c > 0$ such that $|K_t| \leq c\, G^{(m)}_{b,t}$ for all $t > 0$.*

Proof. See [ElR3], Theorem 3.1. □

It also follows from Theorem 2.4 and Example 1.1 that for all $m \in 2\mathbf{N}$ the kernel of the semigroup generated by the operator $(-1)^{m/2} \sum_{i=1}^{d'} A_i^m$ satisfies canonical Gaussian upper bounds for all time if G is nilpotent. Nevertheless in contrast to Theorem 4.1, this result does not extend to Lie groups with polynomial growth, in general. We next describe a counter example.

5. The Euclidean motion group

The Lie algebra $\mathfrak{g}$ of the Euclidean motion group is the three dimensional Lie algebra with basis b_1, b_2, b_3 and commutation relations

$$[b_1, b_2] = b_3 \quad , \quad [b_1, b_3] = -b_2 \quad , \quad [b_2, b_3] = 0 \quad .$$

Then $\mathfrak{g}$ is solvable, but not nilpotent. The maximal nilpotent ideal of $\mathfrak{g}$, the nilradical, equals $\mathfrak{n} = \mathrm{span}(b_2, b_3)$. Let G be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then G is the covering group of the Euclidean motion group. Let a_1, a_2 be an algebraic basis for $\mathfrak{g}$, let $m \in 2\mathbf{N}\backslash\{2\}$ and set

$$H = (-1)^{m/2}(A_1^m + A_2^m) \quad . \tag{4}$$

Let K be the kernel of the semigroup generated by H. One has $V'(\rho) \asymp \rho 3$ for $\rho \geq 1$, so G has polynomial growth.

Theorem 5.1. *The following are equivalent.*

I. *There exist $b, c > 0$ such that*

$$|K_t| \leq c\, G^{(m)}_{b,t}$$

uniformly for all $t > 0$.

II. *There exists a $c \geq 1$ such that*

$$c^{-1}\, V(t)^{-1/m} \leq \|K_t\|_\infty \leq c\, V(t)^{-1/m}$$

uniformly for all $t > 0$.

III. *$a_1 \in \mathfrak{n}$ or $a_2 \in \mathfrak{n}$.*

Proof. See [ElR3] Theorem 1.1 and Remark 2.5. □

A sketch of the beginning of the proof is as follows. Define $\Phi\colon \mathbf{R}3 \to G$ by

$$\Phi(x_1,x_2,x_3) = \exp(x_1 b_1)\exp(x_2 b_2)\exp(x_3 b_3) \quad .$$

Then Φ is a diffeomorphism. Set $B_i = dL(b_i)$ and $\check{B}_i = (\Phi^{-1})_* B_i$. Then

$$\begin{aligned}\check{B}_1 &= -\partial_1,\\ \check{B}_2 &= -\cos x_1\,\partial_2 + \sin x_1\,\partial_3,\\ \check{B}_3 &= -\sin x_1\,\partial_2 - \cos x_1\,\partial_3 \quad ,\end{aligned}$$

where the ∂_i are the partial derivatives on $\mathbf{R}3$. Set $\check{H} = (\Phi^{-1})_* H$. Note that

$$(\varphi, \check{H}\varphi) = \sum_{i=1}^{2} (\check{A}_i^{m/2}\varphi, \check{A}_i^{m/2}\varphi) \tag{5}$$

for all $\varphi \in D(\check{H})$. Then the quadratic form on the right hand side of (5) can be written in the form

$$(\varphi, \check{H}\varphi) = \sum_{\substack{\alpha,\beta\in J(3)\\ 1\leq|\alpha|,|\beta|\leq m/2}} (\partial^\alpha\varphi, c_{\alpha,\beta}\,\partial^\beta\varphi)$$

with the $c_{\alpha,\beta}$ a function which is a polynomial of functions $x \mapsto \sin x_1$ and $x \mapsto \cos x_1$. Hence the quadratic form associated to $\check{H}$ might contain second-order terms. The large time behaviour of the kernel can be obtained using a Bloch–Zak decomposition or by using homogenization theory.

Example 5.2. If $H = B_1 4 + B_2 4$ then K satisfies the canonical Gaussian upper bounds.

If $H = B_1 4 + (B_1 + B_2)4$ then K does not satisfy the canonical Gaussian upper bounds for large time. If one expands $(\check{B}_1 + \check{B}_2)2\varphi$ as in (5) then one obtains on both side of the inner product a term $\check{B}_3\varphi$, so

$$(\varphi, \check{H}\varphi) = (\check{B}_3\varphi, \check{B}_3\varphi) + \text{higher order derivatives} \quad .$$

Similarly to the situation in Theorem 3.2 for sums of subcoercive operators on nilpotent Lie groups one has a contribution of a second-order operator and K does not satisfy the m-th order canonical Gaussian upper bounds for large time.

However, if $H = (B_1 + B_2)4 + B_3 4$ then K does have the canonical Gaussian upper bounds for large time, since Condition III of Theorem 5.1 is valid. This is surprising since the quadratic form has a second-order term contribution as in the second example. But homogenization is a non-linear process and for the homogenized operator in this case the coefficients of the second-order terms cancel. It turns out that the homogenization of $\check{H}$ is again a fourth-order operator and K satisfies the canonical fourth-order Gaussian upper bounds.

Next we describe the behaviour of the kernel in case the equivalent conditions of Theorem 5.1 are not valid.

Theorem 5.3. *Let H be as in* (4)*. Suppose $a_1 \notin \mathfrak{n}$ and $a_2 \notin \mathfrak{n}$. Then one has the following.*

I. *There exist $b, c > 0$ such that*

$$|K_t(g)| \le c \int_N dh\, G^{(m)}_{b,t}(gh^{-1})\, G^{[2]}_{b,t}(h)$$

uniformly for all $g \in G$ and $t \ge 1$, where $N = \exp \mathfrak{n}$ and

$$G^{[2]}_{b,t}(\Phi(0, x_2, x_3)) = t^{-1} e^{-b(x_2 2 + x_3 2)t^{-1}}$$

is the second-order Gaussian on N.

II. *There exists a $c > 0$ such that*

$$c^{-1}\, t^{-(m+1)/m} \le \|K_t\|_\infty \le c\, t^{-(m+1)/m}$$

uniformly for all $t \ge 1$.

III. *There exist $c, c', c_1 > 0$ such that*

$$\|\check{K}_t - \widehat{K}_t\|_\infty \le c\, t^{-(m+1)/m}\, t^{-1/m}$$

uniformly for all $t \ge 1$, where $\check{K}$ is the kernel of the semigroup generated by $\check{H} = (\Phi^{-1})_* H$ and $\widehat{K}$ is the kernel of the semigroup generated by

$$(-1)^{m/2} c_1\, \partial_1^m - c'(\partial_2 2 + \partial_3 2) \quad .$$

Proof. See [ElR3]. □

If $H = (-1)^{m/2} \sum_{|\alpha|=|\beta|=m/2} \partial^\alpha\, c_{\alpha,\beta}\, \partial^\beta$ is a pure m-th order strongly elliptic operator on $\mathbf{R}^d$ with complex measurable coefficients then the solution of the Kato problem solved by Auscher, Hofmann, McIntosh and Tchamitchian states that $D(H^{1/2}) = W^{m/2,2}$ and there exists a $c > 0$ such that

$$c^{-1} \max_{|\alpha|=m/2} \|\partial^\alpha \varphi\|_2 \le \|H^{1/2}\varphi\|_2 \le c \max_{|\alpha|=m/2} \|\partial^\alpha \varphi\|_2 \tag{6}$$

for all $\varphi \in D(H^{1/2})$ (see [AHMT], Theorem 1.5). In [AHMT] the identity $D(H^{1/2}) = W^{m/2,2}$ and the homogeneous estimates (6) were proved first under the additional assumption that the kernel of the semigroup generated by H has the canonical Gaussian upper bounds.

A similar result is valid for subcoercive operators with constant coefficients on nilpotent groups by (3) in Theorem 2.4. If the group is simply connected then an operator as in Theorem 2.4 is unitarily equivalent to an operator on $\mathbf{R}^d$ with polynomial coefficients. If G is a general Lie group and H is a pure m-th order operator which generates a bounded semigroup on L_2 then it follows from Theorem 1.2.V that there exists a $c > 0$ such that

$$
\begin{aligned}
c^{-1}\big(\max_{|\alpha|=m/2} \|A^\alpha\varphi\|_2 + \|\varphi\|_2\big) &\leq \|H^{1/2}\varphi\|_2 + \|\varphi\|_2 \\
&\leq c\,\big(\max_{|\alpha|=m/2} \|A^\alpha\varphi\|_2 + \|\varphi\|_2\big) \qquad (7)
\end{aligned}
$$

for all $\varphi \in D(H^{1/2})$. But if the group is not homogeneous then one cannot easily scale the L_2-norm $\|\varphi\|_2$ of φ away in (7). Nevertheless the homogeneous estimates (3) are valid on nilpotent Lie groups, even if the nilpotent group is not homogeneous.

If G is the covering group of the Euclidean motion group and $H = B_1 4 + B_2 4$, with the notation as in the beginning of this section, then it follows from Theorem 5.1 that the kernel of the semigroup S generated by H has the canonical Gaussian upper bounds. Moreover, S is bounded on L_2 and $D(H^{1/2}) = \bigcap_{|\alpha|=2} D(A^\alpha)$ on L_2. Nevertheless the analogue of the homogeneous estimates (6) and (3) are not valid.

Proposition 5.4. *Let G be the covering group of the Euclidean motion group and adopt the notation as in the beginning of this section. Let a_1, a_2 be an algebraic basis for $\mathfrak{g}$ such that $a_1 \in \mathfrak{n}$ or $a_2 \in \mathfrak{n}$. Let $m \in 2\mathbf{N}\backslash\{2\}$ and set $H = (-1)^{m/2}(A_1^m + A_2^m)$. Then there does* **not** *exist a $c > 0$ such that*

$$\max_{|\alpha|=m/2} \|A^\alpha\varphi\|_2 \leq c\,\|H^{1/2}\varphi\|_2$$

for all $\varphi \in D(H^{1/2})$.

Proof. Suppose there exists a $c > 0$ such that $\max_{|\alpha|=m/2} \|A^\alpha\varphi\|_2 \leq c\,\|H^{1/2}\varphi\|_2$ for all $\varphi \in D(H^{1/2})$. We will show that Condition IV of Theorem 2.3 is valid.

Let K be the kernel of the semigroup S generated by H. Then

$$\max_{|\alpha|=m/2} \|A^\alpha S_t\varphi\|_2 \leq c\,\|H^{1/2}S_t\varphi\|_2 \leq (2e)^{-1/2}t^{-1/2}\|\varphi\|_2$$

for all $t > 0$ by spectral theory. By Theorem 5.1 the kernel K satisfies the canonical Gaussian upper bounds. Hence by a quadrature estimate it follows that there exists a $c_1 > 0$ such that

$$\|S_t\|_{1\to 2} \leq c_1 V'(t)^{-1/(2m)} \quad \text{and} \quad \|K_t\|_2 \leq c_1 V'(t)^{-1/(2m)}$$

for all $t > 0$. Therefore one has

$$\|A^\alpha K_t\|_\infty \le \|A^\alpha S_{3t}\|_{1\to\infty} \le \|A^\alpha S_{2t}\|_{2\to\infty}\|S_t\|_{1\to 2} = \|A^\alpha K_{2t}\|_2\|S_t\|_{1\to 2}$$
$$\le \|A^\alpha S_t\|_{2\to 2}\|K_t\|_2\|S_t\|_{1\to 2} \le (2e)^{-1/2} c\, c_1 2\, t^{-1/2} V'(t)^{-1/m}$$

for all $t > 0$ and $\alpha \in J(d')$ with $|\alpha| = m/2$. Hence there exists a $c_2 > 0$ such that

$$\max_{|\alpha|=m/2} \|A^\alpha K_t\|_\infty \le c_2\, t^{-1/2} V'(t)^{-1/m} \tag{8}$$

for all $t > 0$.

Since H does not have a constant term one has $H\mathbb{1} = 0$ on L_∞, where $\mathbb{1}$ is the constant function with value one. Hence $S_t\mathbb{1} = \mathbb{1}$ and $\int K_t = 1$ for all $t > 0$. Moreover, since H is self-adjoint one deduces that $\overline{K_t(g^{-1})} = K_t(g)$ for all $g \in G$ and $t > 0$. Hence

$$\operatorname{Re} K_{2t}(g) = \operatorname{Re} \int dh\, K_t(h)\, K_t(h^{-1}g)$$
$$\le \int dh\, |K_t(h)|^2 = \int dh K_t(h)\, K_t(h^{-1}) = K_{2t}(e)$$

for all $t > 0$ and $g \in G$ by the Schwartz inequality. By Theorem 5.1 there exist $b, c_3 > 0$ such that $|K_t| \le c_3\, G^{(m)}_{b,t}$ for all $t > 0$. Then for all $\kappa > 0$ one has

$$K_t(e) \ge V'(\kappa t^{1/m})^{-1} \int_{\{g\in G: |g|' \le \kappa t^{1/m}\}} dg\ \operatorname{Re} K_t(g)$$
$$= V'(\kappa t^{1/m})^{-1} \Big(1 - \int_{\{g\in G: |g|' > \kappa t^{1/m}\}} dg\ \operatorname{Re} K_t(g)\Big)$$
$$\ge V'(\kappa t^{1/m})^{-1} \Big(1 - \int_{\{g\in G: |g|' > \kappa t^{1/m}\}} dg\, c_3\, G^{(m)}_{b,t}(g)\Big)$$

for all $t > 0$. But the last integral tends to zero as $\kappa \to \infty$. Hence there exists a $\kappa > 0$ such that $K_t(e) \ge 2^{-1} V'(\kappa t^{1/m})^{-1}$ for all $t > 0$. But since G has polynomial growth there then exists a $c_4 > 0$ such that $K_t(e) \ge c_4\, V'(t)^{-1/m}$ for all $t > 0$.

It follows from a subelliptic variation of Lemma III.3.3 of [Rob] that there exists a $c_5 > 0$ such that

$$\max_{|\alpha|=n} \|A^\alpha \varphi\|_\infty \le \varepsilon^{m/2-n} \max_{|\alpha|=m/2} \|A^\alpha \varphi\|_\infty + c_5\, \varepsilon^{-n} \|\varphi\|_\infty$$

for all $n \in \{1, \ldots, m/2 - 1\}$ and $\varphi \in \bigcap_{|\alpha|=m/2} D(A^\alpha)$ in the L_∞-sense. Hence by (8), using the Gaussian upper bounds for K and choosing

$\varepsilon = t^{-n/m}$ one deduces that

$$\max_{|\alpha|=n} \|A^\alpha K_t\|_\infty \leq (c_2 + c_3 c_5)\, t^{-n/m}\, V'(t)^{-1/m}$$

for all $t > 0$ and $n \in \{1, \ldots, m/2\}$. Then

$$|K_t(g) - K_t(e)| \leq |g|' \Big(\sum_{i=1}^{d'} \|A_i K_t\|_\infty 2 \Big)^{1/2} \leq c_6\, |g|'\, t^{-1/m}\, V'(t)^{-1/m}$$

for all $g \in G$ and $t > 0$, where $c_6 = (d')^{1/2}(c_2 + c_3 c_5)$. It follows that

$$\begin{aligned} c_3 c_4^{-1}\, e^{-b((|g|')^m t^{-1})^{1/(m-1)}} &\geq \left| \frac{K_t(g)}{K_t(e)} \right| \geq 1 - \frac{|K_t(g) - K_t(e)|}{K_t(e)} \\ &\geq 1 - c_6 c_4^{-1}\, |g|'\, t^{-1/m} \end{aligned} \tag{9}$$

for all $g \in G$ and $t > 0$. Next let $\tau \colon \mathbf{C} \to \mathbf{R}$ be a C^∞-function such that $0 \leq \tau \leq 1$, $\tau(z) = 0$ for all $|z| \geq c_3 c_4^{-1} e^{-b}$ and $\tau(z) = 1$ for all $|z| \leq 2^{-1} c_3 c_4^{-1} e^{-b}$. For all $R > 0$ define $\eta_R \in C^\infty(G)$ by

$$\eta_R(g) = \tau\Big(\frac{K_{R^m}(g)}{K_{R^m}(e)} \Big) \quad .$$

Then it follows from (9) that $\eta_R(g) = 0$ if $|g|' \geq R$ and $\eta_R(g) = 1$ if $|g|' \leq \sigma R$ where $\sigma = c_4 c_6^{-1}(1 - 2^{-1} c_3 c_4^{-1} e^{-b})$.

Next we show that the derivatives have the right decay. Let $\alpha \in J(d')$ with $|\alpha| = m/2$. Then

$$(A^\alpha \eta_R)(g) = \sum \tau^{(l)}\Big(\frac{K_{R^m}(g)}{K_{R^m}(e)} \Big) \prod_{p=1}^{l} \frac{(A^{\beta_p} K_{R^m})(g)}{K_{R^m}(e)} \tag{10}$$

uniformly for all $g \in G$ and $R > 0$, where the sum is finite and over a subset of all $l \in \{1, \ldots, n\}$ and $\beta_1, \ldots, \beta_l \in J(d')$ with $|\beta_p| \geq 1$ for all $p \in \{1, \ldots, l\}$ and $|\beta_1| + \ldots + |\beta_l| = n$. Then

$$\left| \prod_{p=1}^{l} \frac{(A^{\beta_p} K_{R2})(g)}{K_{R2}(e)} \right| \leq \prod_{p=1}^{l} (c_2 + c_3 c_5) c_4^{-1} R^{-|\beta_p|} = (c_2 + c_3 c_5)^l c_4^{-l} R^{-n}$$

uniformly for $g \in G$ and $R > 0$. Hence Condition IV of Theorem 2.3 is valid. Therefore by Theorem 2.3 $\mathfrak{g}$ is the direct product of the Lie algebra of a compact group and a nilpotent Lie algebra. This is a contradiction and the proof of the proposition is complete. $\square$

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THE $T(b)$ THEOREM AND ITS VARIANTS

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ABSTRACT. We survey the various local and global variants of $T(1)$ and $T(b)$ theorems which have appeared in their literature, and outline their proofs based on a local wavelet coefficient perspective. This survey is based on recent work in [4] with Pascal Auscher, Steve Hofmann, Camil Muscalu, and Christoph Thiele.

1. INTRODUCTION

The purpose of this expository article is to survey the various types of $T(1)$ and $T(b)$ theorems which have been used to prove boundedness of Calderón-Zygmund kernels. These theorems have many applications to Cauchy integrals and analytic capacity, and also to elliptic and accretive systems; see e.g. [1], [6], [9], [13]. However we will not concern ourselves with applications here, and focus instead on the mechanics of proof of these theorems. In particular we present a slightly non-standard method of proof, based on pointwise estimates of wavelet coefficients.

For simplicity we shall work just on the real line $\mathbb{R}$, with the standard Lebesgue measure dx. For applications one must consider much more complicated settings, for instance one-dimensional sets endowed with Hausdorff measure, but we will not discuss these important generalizations in this article[1], and instead concentrate our attention on the distinction between $T(1)$ and $T(b)$ theorems, and between local and global variants of these theorems.

To avoid technicalities (in particular, in justifying whether integrals actually converge) we shall restrict all functions and operators to be real-valued, and only consider Calderón-Zygmund operators T of the form

$$Tf(x) := \int K(x,y)f(y)\,dy$$

[1]In particular we will not discuss the important recent extensions of the $T(1)$ and $T(b)$ theory to non-doubling measures, see e.g. [24], [23].

where $K(x,y)$ is a smooth, compactly supported kernel obeying the estimates

$$|K(x,y)| \lesssim 1/|x-y| \quad (1)$$

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \lesssim |x-x'|/|x-y|^2 \quad (2)$$

for all distinct x, x', y. Here we use $A \lesssim B$ or $A = O(B)$ to denote the estimate $A \leq CB$ for some constant C. An example of such an operator would be the Hilbert transform $Hf(x) := p.v. \int \frac{1}{x-y} f(y)\, dy$, after first truncating the kernel $p.v. \frac{1}{x-y}$ to be smooth and compactly supported.

Of course, since we have declared K to be smooth and compactly supported, the operator T is *a priori* bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. However, the basic problem is to find useful conditions under which one can obtain more quantitative bounds on the L^p operator norm of T (e.g. depending only on p and the implicit constants in (1), (2) and some additional data, but not on the smoothness and compact support assumptions on K).

The remarkable $T(1)$ theorem of David and Journé [12] gives a complete characterization of this problem:

Theorem 1.1 (Global $T(1)$ theorem)**.** [12] *If T is a Calderón-Zygmund operator such that*

- *(Weak boundedness property) One has $\langle T\chi_I, \chi_I \rangle = O(|I|)$ for all intervals I. Here $|I|$ denotes the length of I.*
- *We have $\|T(1)\|_{BMO} \lesssim 1$.*
- *We have $\|T^*(1)\|_{BMO} \lesssim 1$.*

Then T is bounded on L^p, $1 < p < \infty$

$$\|Tf\|_p \lesssim \|f\|_p$$

(the implicit constants depend on p) and we have the L^∞ to BMO bounds

$$\|Tf\|_{BMO} \lesssim \|f\|_\infty$$
$$\|T^*f\|_{BMO} \lesssim \|f\|_\infty.$$

The remarkable thing about this theorem is that while the conclusion asserts that T and T^* obey certain bounds for all L^p functions f (including when $p = \infty$), whereas the hypotheses only require these type of bounds for very specific L^p functions f. The point is that the kernel bounds (1), (2) impose severe constraints on the behavior of T. Informally, these bounds assert that the bilinear form $\langle Tf, g \rangle$ is small when f and g have widely separated supports, and is especially small when one also assumes that f or g has mean zero. This allows us in many cases to estimate $\langle Tf, g\rangle$ by replacing f with its mean on a certain interval (i.e. by replacing f with its projection onto constant functions), or by performing a similar replacement for g. By doing this at all scales simultaneously (by use of such tools as the wavelet decomposition), we can eventually control $\langle Tf, g\rangle$ by quantities involving $T(1)$, $T^*(1)$, or quantities such as $\langle T\chi_I, \chi_I\rangle$. In more precise terms, the standard proof of Theorem 1.1 proceeds by decomposing T into an easily estimated "diagonal" component (controlled by $\langle T\chi_I, \chi_I\rangle$, and two paraproducts, one related to $T(1)$ and one related to $T^*(1)$, which can then be estimated with the aid of the Carleson embedding theorem. In this note we give an alternate proof (in a model case) based on pointwise estimates of wavelet coefficients.

Despite giving a complete answer to our problem, Theorem 1.1 is not completely satisfactory for two reasons. Firstly, the conditions $\|T(1)\|_{BMO} \lesssim 1$ and $\|T^*(1)\|_{BMO} \lesssim 1$ are global rather than local, in that the constant function 1 and the nature of the BMO norm are both spread out over all intervals of space simultaneously, in a way that the weak boundedness property is not. Thus if one were only interested in whether T is bounded on acertain subset Ω of $\mathbb{R}$, the hypotheses of this theorem would be inappropriate. Secondly, the theorem is rather inflexible in that one must test T and T^* against the function 1, rather than some other functions which might be more convenient to apply T and T^* to.

The first objection is easily addressed. Indeed, it is quite straightforward (and well-known) to see that the global $T(1)$ theorem is equivalent to the following local version:

Theorem 1.2 (Local $T(1)$ theorem)**.** *If T is a Calderón-Zygmund operator such that*

- $\int_I |T\chi_I(x)|^2\, dx \lesssim |I|$ *for all intervals I.*
- $\int_I |T^*\chi_I(x)|^2\, dx \lesssim |I|$ *for all intervals I.*

Then the conclusions of Theorem 1.1 hold.

We sketch the derivation of the global $T(1)$ theorem from the local $T(1)$ theorem as follows. Suppose that $T(1) \in BMO$, then for any interval I we would have

$$\|T(1) - [T(1)]_I\|_{L^2(I)} \lesssim |I|^{1/2},$$

where $[f]_I := \frac{1}{|I|}\int_I f$ denotes the mean of f on I. On the other hand, from (1), (2) and a direct calculation one can show that

$$\|T(1-\chi_I) - [T(1-\chi_I)]_I\|_{L^2(I)} \lesssim |I|^{1/2},$$

and hence that

$$\|T(\chi_I) - [T(\chi_I)]_I\|_{L^2(I)} \lesssim |I|^{1/2}.$$

On the other hand, from the weak boundedness property we have $[T(\chi_I)]_I = O(1)$, and thus we have verified the first property of Theorem 1.2. The second property is verified similarly and we are done. The reverse implication (that the global theorem implies the local version) is similar and we do not detail it here.

As the name suggests, the local $T(1)$ theorem can be localized. Here is one sample localization result:

Theorem 1.3 (Localized $T(1)$ theorem). *Let $\Omega \subset \mathbb{R}$ be an open set. If T is a Calderón-Zygmund operator such that*

- $\int_I |T\chi_I(x)|^2\, dx \lesssim |I|$ *for all intervals I such that $I \cap \Omega \neq \emptyset$.*
- $\int_I |T^*\chi_I(x)|^2\, dx \lesssim |I|$ *for all intervals I such that $I \cap \Omega \neq \emptyset$.*

Then T is bounded on $L^p(\Omega)$ for $1 < p < \infty$, in the sense that

$$\|Tf|_\Omega\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$$

for all $f \in L^p(\Omega)$.

Corresponding L^p and $L^\infty \to BMO$ estimates can also be made, but are a little technical due to the need to define BMO on domains.

We prove this theorem in a model case in Section 3. Unfortunately, this localization is imperfect, since the intervals I in the hypotheses are allowed to partially extend outside Ω; we do not know how to resolve this issue to make the localization more satisfactory.

Now we discuss the second objection to the $T(1)$ theorem, that the choice of function 1 is fixed. Certainly we cannot replace 1 by an arbitrary L^∞ function, since if one replaces 1 by (for instance) the zero function 0 then the hypotheses become trivial, and the theorem absurd. Or if one replaces 1 by a function which vanishes on a large set, then the hypothesis yields us very little information about T on that set, and so one would not expect to conclude boundedness on T inside that set. However, if one replaces 1 by functions which in some sense "never vanish", then one can generalize the $T(1)$ theorem:

Theorem 1.4 (Global $T(b)$ theorem). [13] *Let b, b' be bounded in $L^\infty(\mathbb{R})$ and such that we have the pseudo-accretivity condition*

$$|[b]_I|, |[b']_I| \gtrsim 1$$

for all intervals I. If T is a Calderón-Zygmund operator such that

- *(Modified weak boundedness property) One has $\langle T(b\chi_I), (b'\chi_I)\rangle = O(|I|)$ for all intervals I.*

- *We have* $\|T(b)\|_{BMO} \lesssim 1$.
- *We have* $\|T^*(b')\|_{BMO} \lesssim 1$.

Then T is bounded on L^2

$$\|Tf\|_2 \lesssim \|f\|_2$$

and we have the L^∞ to BMO bounds

$$\|Tf\|_{BMO} \lesssim \|f\|_\infty$$
$$\|T^*f\|_{BMO} \lesssim \|f\|_\infty.$$

The conditions b, b' be bounded in $L^\infty(\mathbb{R})$ can be relaxed slightly to requiring b, b' to just be bounded in BMO; see [4]. This theorem can be proven by using wavelet systems adapted to b, b' respectively [5]. In the special "one-sided" case when $b' = 1$ then there is an alternate proof going through the global $T(1)$ theorem and the heuristic that when $T^*(1) \in BMO$, we have the approximation $\langle T(b), \phi_I\rangle \approx [b]_I \langle T(1), \phi_I\rangle$ for all intervals I and all bump functions ϕ_I of mean zero adapted to I; see [25] (and also [4]).

This $T(b)$ theorem can also be localized, so that instead of having a single function b supported globally, we only need a localized function b_I for each I. Also, each b_I only has to satisfy a single accretivity condition:

Theorem 1.5 (Local $T(b)$ theorem)**.** [4] *If T is a Calderón-Zygmund operator such that*

- *For every interval I, there exists a function $b_I \in L^2(I)$ with*
$$\int_I |b_I|^2 + |Tb_I|^2 \lesssim |I|$$
which obeys the accretivity condition
$$|[b_I]_I| \gtrsim 1.$$
- *For every interval I, there exists a function $b'_I \in L^2(I)$ with*
$$\int_I |b'_I|^2 + |T^*b'_I|^2 \lesssim |I|$$
which obeys the accretivity condition
$$|[b'_I]_I| \gtrsim 1.$$

Then T is bounded on L^2

$$\|Tf\|_2 \lesssim \|f\|_2$$

and we have the L^∞ to BMO bounds

$$\|Tf\|_{BMO} \lesssim \|f\|_\infty$$

$$\|T^*f\|_{BMO} \lesssim \|f\|_\infty.$$

Strictly speaking, the theorem in [4] was only proven in a model case in which "perfect" kernel cancellation conditions were assumed, but the argument extends without significant difficulty to general Calderón-Zygmund kernels. This theorem is a variant of an earlier local $T(b)$ theorem of [9], which was in a much more general setting but required L^∞ conditions on b_I, Tb_I, b'_I, $T^*b'_I$ rather than L^2 conditions. One can also replace the L^2 conditions with L^p and $L^{p'}$ conditions, see [4].

In the "one-sided case", when $T^*1 \in BMO$, then one does not need the second hypothesis (involving the b'_I), and the proof can again proceed via the $T(1)$ theorem, again using the heuristic that when $T^*1 \in BMO$, we have the approximation $\langle Tb_I, \phi_I\rangle \approx [b_I]_I\langle T(1), \phi_I\rangle$ when ϕ_I has mean zero and is adapted to I. (cf. the local $T(b)$ theorems in [6], [1], [2]). However in the general case it appears that one is forced to use tools such as adapted wavelet systems.

2. A dyadic model

In order to simplify the exposition, we shall replace our "continuous" Calderón-Zygmund operators with a "discrete" dyadic model; this model will be much cleaner to manipulate because of the absence of several minor error terms, but will already capture the essence of the arguments.

We define a *dyadic interval* to be any interval of the form $[2^jk, 2^j(k+1)]$ where j, k are integers. We will always ignore sets of measure zero, and so will not distinguish between open, closed, or half-open dyadic intervals. If I is an interval, we use $|I|$ to denote its length and I^{left} and I^{right} to denote left and right halves of I respectively, and refer to I^{left} and I^{right} as *siblings*. We define the *Haar wavelet* ϕ_I to be the L^2-normalized function

$$\phi_I := |I|^{-1/2}(\chi_{I^{left}} - \chi_{I^{right}}).$$

As is well-known, these wavelets form an orthonormal basis of $L^2(\mathbb{R})$. For any locally integrable f, we define the *wavelet transform* Wf, defined on the space of dyadic intervals, by $Wf(I) := \langle f, \phi_I\rangle$. Thus W is an isometry from L^2 to l^2.

Observe that if T obeys (1), (2), and I and J are disjoint dyadic intervals, then the quantity $\langle T\phi_I, \phi_J\rangle$ decays quickly in the separation of I and J (for instance, we have the estimates

$$\langle T\phi_I, \phi_J\rangle \lesssim \frac{|I|}{|J|}\left(1 + \frac{\text{dist}(I, J)}{|J|}\right)^{-2}$$

when $|J| \geq |I|$). We now introduce a simplified model operator in which the quantity $\langle T\phi_I, \phi_J\rangle$ not only decays, but in fact *vanishes* when I, J are disjoint.

Definition 2.1. *A perfect dyadic Calderón-Zygmund operator T is an operator of the form*

$$Tf(x) := \int K(x,y)f(y)\, dy$$

where $K(x,y)$ is a bounded, compactly supported function which obeys the kernel condition

$$|K(x,y)| \lesssim \frac{1}{|x-y|} \tag{3}$$

and the perfect dyadic Calderón-Zygmund conditions

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| = 0 \tag{4}$$

whenever $x, x' \in I$ and $y \in J$ for some disjoint dyadic intervals I and J. Equivalently, K is constant on all rectangles $\{I \times J : I, J \text{ are siblings}\}$.

As a consequence of (4) we see that Tf_I is supported on I whenever f_I is supported on I with mean zero, and similarly with T replaced by T^*. In particular we have $\langle T\phi_I, \phi_J\rangle = 0$ whenever I, J are disjoint.

We define the dyadic BMO norm by

$$\|f\|_{BMO_\Delta} := \sup_I \left(\frac{1}{|I|}\int_I |f - [f]_I|^2\right)^{1/2} = \sup_I \left(\sum_{J\subseteq I} Wf(J)^2\right)^{1/2}$$

where I ranges over all dyadic intervals. This norm is very close to the usual BMO norm, see [18].

3. Pointwise wavelet coefficient estimates

The purpose of this section is to give a somewhat non-standard proof of the $T(1)$ theorem in the model case of perfect dyadic Calderón-Zygmund operators. This differs from the usual proof (involving paraproduct decomposition and Carleson embedding) in that it relies on pointwise estimates for wavelet coefficients, and also tackles the endpoint $L^\infty \to BMO$ estimates first (instead of the usual procedure of first establishing L^2 bounds.

Let T be a perfect dyadic Calderón-Zygmund operator, and let $f \in L^2(\mathbb{R})$ be a function. In order to study the L^2 or BMO norm of Tf, it makes sense to study the wavelet coefficients $W(Tf)(J)$. It turns out that one can write the coefficients of Tf rather explicitly in terms of the corresponding coefficients of f, $T\chi_J$, $T^*\chi_J$ and $fT^*\chi_J$:

Lemma 3.1. [4] *We have*

$$\begin{aligned} W(Tf)(J) = {} & [f]_J W(T\chi_J)(J) - [f]_J W(T^*\chi_J)(J) + W(fT^*\chi_J)(J) \\ & - \frac{2}{|J|}(\langle T\chi_{J_{left}}, \chi_{J_{right}}\rangle + \langle T\chi_{J_{right}}, \chi_{J_{left}}\rangle) Wf(J). \end{aligned} \quad (5)$$

From (3) we thus have the useful pointwise estimate

$$\begin{aligned} |W(Tf)(J)| \lesssim {} & |[f]_J||W(T\chi_J)(J)| + |[f]_J||W(T^*\chi_J)(J)| \\ & + |W(fT^*\chi_J)(J)| + |Wf(J)| \end{aligned} \quad (6)$$

on the wavelet coefficient of Tf. The pointwise estimates (6) are reminiscent of the standard pointwise sharp function estimates $(Tf)^\#(x) \lesssim Mf(x)$ for Calderón-Zygmund kernels bounded on L^p; see [26].

Proof First observe that if f is vanishes on J then both sides of (5) vanish (since $W(Tf)(J) = \langle f, T^*\phi_J\rangle$ and $T^*\phi_J$ is supported on J). Thus by linearity we may assume that f is supported on J.

Now suppose that f is equal to χ_J, then $Wf(J) = 0$, and (5) collapses to

$$W(T\chi_J)(J) = W(T\chi_J)(J) - W(T^*\chi_J)(J) + W(\chi_J T^*\chi_J)(J)$$

which is clearly true. By linearity we may thus assume that f is orthogonal to χ_J.

Now suppose that f is equal to ϕ_J. Then (5) simplifies to

$$\langle T\phi_J, \phi_J\rangle = \frac{1}{|J|}\langle T\chi_J, \chi_J\rangle - \frac{2}{|J|}(\langle T\chi_{J_{left}}, \chi_{J_{right}}\rangle + \langle T\chi_{J_{right}}, \chi_{J_{left}}\rangle),$$

which is easily verified by expanding out T. By linearity we may thus assume that f is also orthogonal to ϕ_J. In particular, we can now assume that f has mean zero on both J_{left} and J_{right}. After some re-arranging, the claim now collapses to

$$\langle Tf, \phi_J\rangle = \langle T(f\phi_J), \chi_J\rangle.$$

If f is supported on J_{left}, then the claim follows since ϕ_J is constant on the support of f and on Tf. Similarly if f is supported on J_{right}. The claim now follows by linearity. ∎

Of course, this identity took full advantage of the perfect cancellation (4). However in the continuous case there are still analogues of (6), but with some additional error terms which are easily treated. (Basically, the right-hand side will not just contain terms related to J, but also terms related to intervals J' which are "close" in size and location to J, but with an additional weight which decays fairly quickly as J' moves further away from J). One can also replace the Haar wavelet with smoother wavelets in order to avoid certain (harmless) logarithmic divergences which can come up in this procedure. We will not pursue the details.

With (6) in hand it is a fairly easy matter to prove the dyadic analogue of Theorem 1.2:

Theorem 3.2 (Local dyadic $T(1)$ theorem)**.** *If T is a perfect dyadic Calderón-Zygmund operator such that*

- $\int_I |T\chi_I(x)|^2\, dx \lesssim |I|$ *for all dyadic intervals I.*
- $\int_I |T^*\chi_I(x)|^2\, dx \lesssim |I|$ *for all dyadic intervals I.*

Then T is bounded on L^2

$$\|Tf\|_2 \lesssim \|f\|_2$$

and we have the L^∞ to dyadic BMO bounds

$$\|Tf\|_{BMO_\Delta} \lesssim \|f\|_\infty$$

$$\|T^*f\|_{BMO_\Delta} \lesssim \|f\|_\infty.$$

Of course, this local theorem is equivalent to its global counterpart by the argument sketched in the introduction.

Proof First we show that T maps L^∞ into dyadic BMO. Fix $f \in L^\infty$; we may normalize $\|f\|_\infty = 1$. We need to show the Carleson measure condition

$$\sum_{J\subseteq I} |W(Tf)(J)|^2 \lesssim |J| \tag{7}$$

for all dyadic intervals I.

By a standard iteration procedure (extremely common in the study of Carleson measures or BMO; we will use it again in the next section) we may restrict the summation to those intervals J for which

$$|[T^*\chi_I]_J| \leq C \tag{8}$$

for some large constant C. We sketch the reason for this is as follows. By Cauchy-Schwartz, we see that $\int_J |T^*\chi_I|^2 \geq C^2|J|$ whenever (8) fails. Since we are assuming the bound $\int_I |T^*\chi_I|^2 \lesssim |I|$, the intervals J for which (8) fails can therefore only cover a set of measure at most $\frac{1}{100}|I|$, if C is chosen sufficiently large. One can then iterate away those contributions in the usual manner (e.g. by assuming inductively that (7) already holds for all sub-intervals of I).

Fix I; we will implicitly assume that (8) holds. Now we use (6) to decompose the wavelet coefficients $W(Tf)(J)$.

We write

$$W(fT^*\chi_J)(J) = W(fT^*\chi_I)(J) - W(fT^*\chi_{I\setminus J})(J).$$

From (4) we see that $T^*\chi_{I\backslash J}$ is constant on J, thus

$$W(fT^*\chi_{I\backslash J})(J) = ([T^*\chi_I]_J - [T^*\chi_J]_J)Wf(J).$$

From the hypotheses and Cauchy-Schwarz we have $[T^*\chi_J]_J = O(1)$. From this and (8) we thus have

$$|W(fT^*\chi_J)(J)| \lesssim |W(fT^*\chi_I)(J)| + |Wf(J)|.$$

Also, since $T^*\phi_J$ is supported on J, we have

$$W(T\chi_J)(J) = W(T\chi_I)(J),$$

and similarly for T^*. Since $[f]_J = O(\|f\|_\infty) = O(1)$, we thus see from (6) that we have the pointwise estimate

$$\begin{aligned}|W(Tf)(J)| \lesssim\, & |W(T\chi_I)(J)| + |W(T^*\chi_I)(J)| \\ & + |W(fT^*\chi_I)(J)| + |W(f)(J)|.\end{aligned}$$

By hypothesis, the functions $T\chi_I$, $T^*\chi_I$, $(T^*\chi_I)f$, and f all have an $L^2(I)$ norm of $O(|I|^{1/2})$. The claim (7) follows since the ϕ_J are orthonormal in $L^2(I)$. This shows that T maps L^∞ into BMO_Δ.

A similar argument shows that T^* maps L^∞ to BMO_Δ. One then concludes L^p boundedness by Fefferman-Stein interpolation (see e.g. [26]) and a standard reiteration argument (starting, for instance, with the apriori L^2 boundedness and then improving this by repeated interpolation; cf. [29]). Alternatively, one could use L^p Calderón-Zygmund techniques as in [16] to first establish weak L^p bounds and then interpolate. We omit the details ∎

We now sketch how the above theorem localizes to a domain $\Omega \subset \mathbb{R}$. Specifically, we prove

Corollary 3.3 (Dyadic localized $T(1)$ theorem)**.** *Let $\Omega \subset \mathbb{R}$ be an open set. Let $\mathbf{I}$ be the set of dyadic intervals I such that $I \cap \Omega \neq \emptyset$. If T is a Calderón-Zygmund operator such that*

- $\int_I |T\chi_I(x)|^2\, dx \lesssim |I|$ *for all $I \in \mathbf{I}$.*
- $\int_I |T^*\chi_I(x)|^2\, dx \lesssim |I|$ *for all $I \in \mathbf{I}$.*

Then T is bounded on $L^p(\Omega)$ for $1 < p < \infty$, in the sense that

$$\|Tf|_\Omega\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$$

for all $f \in L^p(\Omega)$.

Proof Let Π be the projection

$$\Pi f := \sum_{I \in \mathbf{I}} Wf(I)\phi_I.$$

Equivalently, we have $\Pi f(x) = f(x)$ if $x \in \Omega$, and $\Pi f(x) = [f]_I$ when $x \notin \Omega$, where I is the smallest dyadic interval containing x and intersecting Ω.

We consider the operator $\Pi T \Pi$. It is easy to verify that this operator is also a perfect dyadic Calderón-Zygmund operator. We claim that $\Pi T \Pi$ verifies the hypotheses of Theorem 3.3. The claim then follows since $\Pi T \Pi f(x) = T \Pi f(x) = Tf(x)$ for all $x \in \Omega$ and $f \in L^p(\Omega)$.

Let I be a dyadic interval. We shall verify that

$$\int_I |\Pi T \Pi \chi_I(x)|^2 \, dx \lesssim |I|.$$

First suppose that $I \in \mathbf{I}$. Then $\Pi \chi_I = \chi_I$. Also, on I one can estimate Π by the Hardy-Littlewood maximal function on I. The claim then follows from the hypotheses and the Hardy-Littlewood maximal inequality.

Now suppose that $I \notin \mathbf{I}$. Let J be the smallest dyadic interval in $\mathbf{I}$ which contains J. Then $\Pi \chi_I = \frac{|I|}{|J|} \chi_J$, and $\Pi f = [f]_J$ on I. Thus it will suffice to show that

$$\frac{1}{|J|} \int_J \left| \frac{|I|}{|J|} T \chi_J \right|^2 \lesssim |I|.$$

But this follows from hypothesis since $|I| \leq |J|$. ∎

It is also possible to establish the above corollary directly by using variants of the pointwise estimate (6), combined with the square-function wavelet characterization of L^p and some Calderón-Zygmund theory, but we will not detail this here.

The operator Π is an example of a *phase space projection*, to a region of phase space known as a "tree". Such localized estimates for Calderón-Zygmund operators are very useful in the study of multilinear objects such as the bilinear Hilbert transform and the Carleson maximal operator: see e.g. [15], [28], [19], [20], [22], etc. (For a further discussion of the connection between Carleson measures, Calderón-Zygmund theory, and phase space analysis, see [4]).

3.1. **Adapted Haar bases, and $T(b)$ theorems.** We now turn to $T(b)$ theorems, starting with the global $T(b)$ theorem in Theorem 1.4. We shall consider the dyadic model case when T is a perfect dyadic Calderón-Zygmund operator, and assume that b, b' are bounded and obey the pseudo-accretivity conditions

$$|[b]_I|, |[b']_I| \gtrsim 1$$

for all dyadic intervals I.

Ideally, one would like to have an estimate like (6), but with $T(\chi_J)$ and $T^*(\chi_J)$ replaced by $T(b\chi_J)$ and $T^*(b'\chi_J)$:

$$|W(Tf)(J)| \lesssim |[f]_J||W(T(b\chi_J))(J)| + |[f]_J||W(T^*(b'\chi_J))(J)| \\ + |W(fT^*(b'\chi_J))(J)| + |Wf(J)|. \tag{9}$$

This is because one can then approximate $T(b\chi_J)$ and $T^*(b'\chi_J)$ by $T(b)$ and $T^*(b')$ by arguments similar to that used to prove Theorem 3.2.

Unfortunately, such an estimate does not seem to hold in general. However, if one replaces the Haar wavelet system by modified Haar wavelet systems adapted to b and b', then one can recover a formula similar to (9), as follows.

For each interval I, we define the adapted Haar wavelet ϕ_I^b (introduced in [10]; see also [5]) by

$$\phi_I^b := \phi_I - \frac{Wb(I)}{[b]_I}\frac{\chi_I}{|I|}. \tag{10}$$

The wavelet ϕ_I^b no longer has mean zero, but still obeys the weighted mean zero condition

$$\int b\phi_I^b = 0. \tag{11}$$

As a consequence we see that

$$\int \phi_I^b b \phi_J^b = 0 \text{ for all } I \neq J. \tag{12}$$

A calculation gives the identity

$$\int \phi_I^b b \phi_I^b = \frac{2}{[b]^{-1}_{I_{left}} + [b]^{-1}_{I_{right}}}.$$

From the pseudo-accretivity and boundedness properties, we thus see that ϕ_P^b has the non-degeneracy property

$$\left| \int \phi_P^b b \phi_P^b \right| \sim 1. \tag{13}$$

Define the dual adapted Haar wavelet ψ_I^b by

$$\psi_I^b := \frac{\phi_I^b b}{\int \phi_I^b b \phi_I^b}.$$

By (12), (13) we thus have that $\langle \psi_I^b, \phi_J^b \rangle = \delta_{IJ}$ where δ is the Kronecker delta. In particular we have the representation formula

$$f = \sum_I W_b f(I) \psi_I^b \tag{14}$$

(formally, at least), where the adapted wavelet coefficients $W_b f(I)$ are defined by

$$W_b f(I) := \langle f, \phi_I^b \rangle.$$

From some computation and the Carleson embedding theorem one can eventually verify the orthogonality property

$$\Big(\sum_I |W_b f(I)|^2 \Big)^{1/2} \sim \|f\|_2 \tag{15}$$

(see e.g. [26], or [4]). Thus the adapted wavelet transform W_b has most of the important properties that W has.

The analogue of (9) is

Lemma 3.4. [4] *If f is bounded in L^∞, then*

$$|W_b(bT^* f)(I)| \lesssim |W_b(bT^*(b'\chi_I))(I)| + |W_b(b'T(b\chi_I))(I)| \\ + |W_b(fT(b\chi_I))(I)| + |Wf(I)| + |W(b')(I)|.$$

Proof We can write

$$Wb(bT^* f)(I) = \langle f, T(b\phi_I^b) \rangle = \langle f, T\psi_I^b \rangle.$$

Since ψ_I^b has mean zero, $T\psi_I^b$ is supported on I. Thus the left-hand side does not depend on the values of f outside I. Similarly for the right-hand side. Thus we may assume that f is supported on I.

The claim is clearly true when f is equal to a bounded multiple of $b'\chi_I$ (since the left-hand side is then bounded by the first term on the right-hand side). Since $|[b']_I| \sim 1$, every bounded function f on I can be split as the sum of a bounded multiple of $b'\chi_I$ and a bounded function on I with mean zero. Thus it will suffice to prove the estimate

$$|W_b(bT^* f)(I)| \lesssim |W_b(fT(b\chi_I))(I)| + |Wf(I)|$$

for functions f of mean zero on I.

We can write

$$W_b(bT^* f)(I) = \langle \phi_I^b T^*(f), b\chi_I \rangle.$$

Since

$$\langle T^*(\phi_I^b f), b\chi_I \rangle = \langle fT(b\chi_I), \phi_I^b \rangle = W_b(fT(b\chi_I))(I)$$

it will suffice to prove the commutator estimate

$$\langle \phi_I^b T^*(f), b\chi_I \rangle - \langle T^*(\phi_I^b f), b\chi_I \rangle = O(|Wf(I)|).$$

Recall that ϕ_I^b is constant on I_{left} and I_{right}. Thus if f is supported on I_{left} with mean zero, then the commutator vanishes (since $T^*(\phi_I^b f) = \phi_I^b(I_{left})T^*(f)$ is supported on I_{left}). Similarly if f is supported on I_{right} with mean zero. Since we are already assuming that f is supported on

I with mean zero, it thus suffices to verify the estimate when f is the standard Haar wavelet ϕ_I:

$$\langle \phi_I^b T^*(\phi_I), b\chi_I\rangle - \langle T^*(\phi_I^b \phi_I), b\chi_I\rangle = O(1).$$

Throwing the T^* on the other side, we see the claim will follow from Cauchy-Schwarz and the estimates

$$\int_I |T(b\chi_{I_{left}})|^2, \int_I |T(b\chi_{I_{right}})|^2, \int_I |T(b\chi_I)|^2 \lesssim |I|.$$

We just show the last estimate, as the first two are proven similarly. From weak boundedness we have

$$\Big|\int_I T(b\chi_I)b'\Big| \lesssim |I|^{1/2}.$$

Since b' is in L^∞ with $|[b']_I| \sim 1$, it thus suffices to prove the BMO estimate

$$\int_I |T(b\chi_I) - [T(b\chi_I)]_I|^2 \lesssim |I|.$$

On the other hand, since $T(b)$ is in BMO by hypothesis, we have

$$\int_I |T(b) - [T(b)]_I|^2 \lesssim |I|.$$

The claim follows since $T(b(1-\chi_I))$ is constant on I by (4). ∎

From Lemma 3.4 one can show (as in the proof of Theorem 1.2) that the wavelet coefficients $W_b(bT^*f)(I)$ obeys the Carleson measure estimate

$$\sum_{J\subseteq I} |W_b(bT^*f)(J)|^2 \lesssim |I|.$$

(As in the proof of Theorem 1.2, it will be convenient to first remove the intervals where the mean of $T(b\chi_I)$ is large, in order to estimate $T(b\chi_J)$ by $T(b\chi_I)$ effectively.)

From this Carleson measure estimate, (15) and the mean-zero property of the wavelets $\psi_J^b = b\phi_J^b$, we can show that $T^*f \in BMO$. Thus T^* maps L^∞ to BMO. Similar arguments give the bound for T, and the L^p bounds then follow from interpolation as before. This allows one to prove the global $T(b)$ theorem, Theorem 1.4.

We now briefly indicate how the above proof of the global $T(b)$ theorem can be localized to yield Theorem 1.5; the details though are rather complicated and can be found in [4]. In the above argument we gave pointwise estimates on wavelet coefficients $W_b(bT^*f)(J)$, which were in turn used to prove Carleson measure estimates. Now, however, we do not have a single b to work with, instead having a localized b_I assigned to each interval I. We would now like to have a Carleson measure estimate of the form

$$\sum_{J \subseteq I} |W_{b_I}(b_I T^* f)(J)|^2 \lesssim |I|. \tag{16}$$

Suppose for the moment that we could attain (16). If b_I obeyed the pseudo-accretivity condition $|[b_I]_J| \gtrsim 1$ for all $J \subseteq I$, and the boundedness condition $\int_J |b_I|^2 \lesssim |J|$ for all $J \subseteq I$, then one could use (15) to then show the BMO estimate

$$\int_I |T^* f - [T^* f]_I|^2 \lesssim |I| \tag{17}$$

which would show that T^* mapped L^∞ to BMO as desired.

Unfortunately, our assumptions only give us that b_I has large mean on all of I:

$$|[b_I]_I| \gtrsim 1.$$

This does not preclude the possibility of b_I having small mean on some sub-interval J of I (for instance, b_I could vanish on some sub-interval). However, because we have the L^2 bound $\int_I |b_I|^2 \lesssim I$, it does mean that $[b_I]_J$ cannot vanish for a large proportion of intervals J. To make this more rigorous, fix $0 < \varepsilon \ll 1$, and let $\Omega_I \subseteq I$ be the union of all the intervals $J \subseteq I$ for which $|[b_I]_J| \lesssim \varepsilon$. Then by a simple covering argument (splitting Ω_I as the disjoint union of the maximal dyadic intervals J in Ω_I) we have

$$\Big| \int_{\Omega_I} b_I \Big| \lesssim \varepsilon |\Omega_I| \lesssim \varepsilon |I|$$

and thus (if ε is sufficiently small)

$$\Big| \int_{I \backslash \Omega_I} b_I \Big| \gtrsim |I|.$$

From Cauchy-Schwarz and the L^2 bound on b_I we thus have

$$|I \backslash \Omega_I| \gtrsim |I|.$$

Thus there is a significant percentage of I for which one does not have the problem of b_I having small mean. Furthermore, by a similar argument one can also show that on a slightly smaller significant percentage of I, one also does not have the problem of $\int_J |b_I|^2$ being much larger than J. Thus there is a non-zero "good" portion of I where the function b_I behaves as if itwere pseudo-accretive, and on which Carleson estimates such as (16) would yield BMO bounds. The proof of (17) then proceeds by first estimating the integral on this "good" portion of I, and using a standard iteration argument to deal with the remaining "bad" portion of I, which has measure at most $c|I|$ for some $c < 1$; the point being that the geometric series $1 + c + c^2 + \dots$ converges. Details can be found in [4]; similar arguments can also be found in [1], [6] (and indeed a vector-valued, higher-dimensional version of this trick of removing the small-mean intervals is crucial to the resolution of the Kato square root problem in higher dimensions [1], [2]).

It remains to prove (16), at least when J is restricted to the set where b_I behaves like a pseudo-accretive function. Obviously one would like to have Lemma 3.4 holding (but with b, b' replaced by b_I, b'_I of course). However, an inspection of the proof shows that to do this one needs bounds of the form $\int_J |b_I|^2 \lesssim |J|$, $\int_J |b'_I|^2 \lesssim |J|$, and $|[b'_I]_J| \gtrsim 1$; in other words b_I and b'_I have to behave like pseudo-accretive functions[2] on J. Fortunately, by arguments similar to the ones given above, one can show that the bad intervals J, for which the above bounds fail, do not cover all of I, and there is a fixed proportion of I which is "good", in the sense that (16) holds when localized to this good set. One then performs yet another iteration argument on the "bad" portion of I to conclude (16).

This concludes the sketch of the proof of Theorem 1.5 in the case of perfect dyadic Calderón-Zygmund operators; further details can be found in [4]. It is an interesting question as to whether this proof technique can also produce localized estimates, perhaps similar to Theorem 1.3. One model instance of such a theorem is Theorem 29 of [9], which says informally that if E is a compact subset of an Ahlfors-David regular set with positive analytic capacity, then E has large intersection with another Ahlfors-David regular set for which the Cauchy integral is bounded. (This is somewhat analogous with a hypothesis that $T(b), T^*(b) \in BMO$ for some b which is sometimes, but not always, pseudo-accretive, and a conclusion that T is bounded when restricted to a reasonably large set Ω).

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[2]Actually, one also needs lower bounds for the magnitude of of $[b'_I]_{J_{left}}$ and $[b'_I]_{J_{right}}$. This introduces some technicalities involving "buffer" intervals - intervals J for which b_I has large mean, but such that b_I has small mean on one of the subintervals J_{left}, J_{right}. These intervals are rather annoying to deal with but are fortunately not too numerous; see [4].

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SEPARATION OF GRADIENT YOUNG MEASURES AND THE BMO

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Dedicated to Professor Alan McIntosh for his 60th birthday

Abstract. Let $K = \{A, B\} \subset M^{N\times n}$ with $\mathrm{rank}(A - B) > 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded arcwise connected Lipschitz domain. We show that there is a direct estimate of the size of the ϵ-neighborhood K_ϵ of K such that $K_\epsilon = \bar{B}_\epsilon(A) \cup \bar{B}_\epsilon(B)$ separates gradient Young measures, that is, if $(u_j) \subset W^{1,1}(\Omega, \mathbb{R}^N)$ is bounded and $\int_\Omega \mathrm{dist}(Du_j, K_\epsilon)dx \to 0$ as $j \to \infty$, then up to a subsequence, either $\int_\Omega \mathrm{dist}(Du_j, \bar{B}_\epsilon(A))dx \to 0$ or $\int_\Omega \mathrm{dist}(Du_j, \bar{B}_\epsilon(B))dx \to 0$.

In this note I present some direct estimate on neighborhoods K_ϵ of a two matrix set $K = \{A, B\}$ that separate gradient Young measures. I also show how one can use the BMO seminorm of approximate solutions of linear elliptic systems to control the oscillation of sequences gradients approaching K_ϵ. The note is based on a recent work [19] with a slightly different approach. In the case of the two matrix set, we can establish our main result (Theorem 1 below) without quoting a recent deep approximation theorem obtained by Müller [14], instead, most of the tools we use will be standard in the calculus of variations. The problem we consider is motivated from the variational approach of material microstructure [4, 5], in particular, the study of metastablility, hysteresis and numerical analysis related to them.

Let $M^{N\times n}$ be the space of $N \times n$ real matrices with $N, n \geq 2$. Let Ω be a bounded arcwise connected Lipschitz domain throughout this note. We denote by $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ weak convergence and weak-$*$ convergence respectively. The characteristic function of a set $V \subset \mathbb{R}^n$ is denoted by χ_V. The following result is now well-known [4, 17].

Theorem A. *Let $A, B \in M^{N\times n}$ with $\mathrm{rank}(A - B) > 1$. Let $(u_j) \subset W^{1,p}(\Omega, \mathbb{R}^N)$ $(1 \leq p < \infty)$ be a bounded sequence such that $\lim_{j\to\infty} \int_\Omega \mathrm{dist}^p(Du_j, \{A, B\}) \to 0$. Then, up to a subsequence either $Du_j \to A$ a.e. or $Du_j \to B$ a.e..*

In the study of metastablility and hysteresis of material microstructure, Ball and James [6] established the following:

Theorem B. *If a compact set* $K = K_1 \cup K_2 \subset M^{N\times n}$ *(*$K_1 \cap K_2 = \emptyset$*) separates gradient Young measures in the sense that* $\operatorname{supp}\nu_x \subset K$ *a.e.* $\Rightarrow \operatorname{supp}\nu_x \subset K_1$ *or* $\operatorname{supp}\nu_x \subset K_2$ *a.e. Then there exists* $\epsilon > 0$*, such that* K_ϵ *still separates gradient Young measures.*

In other words, K separates gradient Young measures if $\operatorname{dist}(Du_j, K) \to 0$ in $L^1(\Omega)$ implies, up to a subsequence either $\operatorname{dist}(Du_j, K_1) \to 0$ in $L^1(\Omega)$ or $\operatorname{dist}(Du_j, K_2) \to 0$ in $L^1(\Omega)$. Theorem B then claims that there exists $\epsilon > 0$ such that K_ϵ still separates gradient Young measures.

Theorem B was established by using a contradiction argument in [6]. In this note I give an estimate of $\epsilon > 0$ for the special case $K = \{A, B\}$ with $\operatorname{rank}(A-B) > 1$ such that $K_\epsilon = \bar{B}_\epsilon(A) \cup \bar{B}_\epsilon(B)$ separates gradient Young measures, i.e. we give an estimate of the size of the balls such that a sequence of gradients approaching two balls can only approach one. We have,

Theorem 1. *Let* $K = \{A, B\} \subset M^{N\times n}$ *with* $\operatorname{rank}(A-B) > 1$*. Let* $K_\epsilon = \bar{B}_\epsilon(A) \cup \bar{B}_\epsilon(B)$*, and* $\lambda_{\max}$ *be the largest eigenvalue of* $(A-B)^t(A-B)/|A-B|^2$*. Then there exists an estimate of* $\epsilon > 0$ *depending on* n*,* N*,* $|A-B|$*,* $\lambda_{\max}$ *such that* $u_j \rightharpoonup u$ *in* $W^{1,1}(\Omega, \mathbb{R}^N)$*,* $\int_\Omega \operatorname{dist}(Du_j, K_\epsilon)dx \to 0$ *as* $j \to \infty$ *implies that up to a subsequence, either*

$$\lim_{j\to\infty} \int_\Omega \operatorname{dist}(Du_j, \bar{B}_\epsilon(A))dx = 0, \quad \textit{or} \quad \lim_{j\to\infty} \int_\Omega \operatorname{dist}(Du_j, \bar{B}_\epsilon(B))dx = 0.$$

In the language of gradient Young measures [11, 15], this means $\operatorname{supp}\nu_x \subset K_\epsilon$, a.e. $x \in \Omega$ implies either $\operatorname{supp}\nu_x \subset \bar{B}_\epsilon(A)$ a.e. or $\operatorname{supp}\nu_x \subset \bar{B}_\epsilon(B)$ a.e.

We use the following standard tools to establish Theorem 1.

(i) A non-negative quasiconvex function vanishing exactly on K_ϵ [10, 17], (ii) homogeneous Young measures [15], (iii) BMO estimates for elliptic systems [9], (iv) Besicovitch's covering lemma [8].

More precisely, we have an explicit non-negative quasiconvex function f satisfying

$$0 \le f(X) \le C(1 + |X|^2), \qquad f^{-1}(0) = K_\epsilon.$$

So if $\int_\Omega \operatorname{dist}^2(Du_j, K_\epsilon)dx \to 0$ and $u_j \rightharpoonup u$ in $W^{1,2}$, then by the well-known weak lower semicontinuity theorem of Acerbi and Fusco [1],

$$0 = \liminf_{j\to\infty} \int_\Omega f(Du_j)dx \ge \int_\Omega f(Du)dx.$$

In the language of gradient Young measures, this gives

$$\int_\Omega \int_{M^{N\times n}} f(\lambda) d\nu_x(\lambda) dx = \int_\Omega f(\bar\nu_x) dx, \quad \text{where } \bar\nu_x = \int_{M^{N\times n}} \lambda d\nu_x = Du(x),$$

hence $\operatorname{supp}\nu_x \subset K_\epsilon$, and more importantly, we can **locate** the weak limit

$$\bar\nu_x = Du(x) \in K_\epsilon \text{ a.e.} \tag{1}$$

Recall that a continuous function $f : M^{N\times n} \to \mathbb{R}$ is quasiconvex [13, 3] if

$$\int_\Omega f(A + D\phi) \geq f(A)|\Omega|, \quad \text{for } A \in M^{N\times n},\ \phi \in C_0^1(\Omega,\, \mathbb{R}^N).$$

It is well known that if $0 \leq f(X) \leq C(1+|X|^p)$, the variational integral $u \to \int_\Omega f(Du)dx$ is sequentially weakly lower semicontinuous in $W^{1,p}$ if and only if f is quasiconvex [1].

We may construct quasiconvex functions by calculating the quasiconvex envelope QF for a given function $F : M^{N\times n} \to \mathbb{R}$: $QF = \sup\{g \leq F,\ g \text{ quasiconvex}\}$. In our case, we let $F(X) = \operatorname{dist}^2(X,\, \{A,\, B\})$, then QF can be explicitly calculated [10] and

$$\begin{aligned} QF(X) = F(X) &= \operatorname{dist}^2(X,\, \{A,\, B\}), \\ &\text{if} \quad \operatorname{dist}^2(X,\, \{A,\, B\}) \leq |A-B|^2(1-\lambda_{\max})/4. \end{aligned}$$

Let

$$f(X) = \left[Q\operatorname{dist}^2(X,\, \{A,\, B\}) - \epsilon\right]_+, \qquad \epsilon \leq \frac{|A-B|^2(1-\lambda_{\max})}{4},$$

then $f \geq 0$ is quasiconvex and $f^{-1}(0) = K_\epsilon$.

Next need the $W^{1,\infty}$-Gradient Young measure and the homogeneous Young measure [11, 15] to **localize** our problem: We only need a special case of the general theorem of gradient Young measures. Let $K \subset M^{N\times n}$ be compact, $u_j \rightharpoonup u$ in $W^{1,1}(\Omega,\, \mathbb{R}^N)$ such that $\operatorname{dist}(Du_j,\, K) \to 0$ in L^1. Then

(I) up to a subsequence, there exists a family of probability measures (gradient Young measures) $\operatorname{supp}\nu_x \subset K$, a.e. $x \in \Omega$, $f(Du_j) \rightharpoonup \int_K f(\lambda)d\nu_x$ weakly in L^1, for all continuous functions f satisfying $|f(X)| \leq C(|X|+1)$ and the weak limit can be identified as $Du(x) = \int_K \lambda d\nu_x := \bar\nu_x$ a.e. $(\nu_x)_{x\in\Omega}$ is called the family of $W^{1,\infty}$-gradient Young measures generated by (a subsequence) of gradients (Du_j).

(II) There is a bounded sequence $v_j \in W^{1,\infty}$ such that (Dv_j) generates the same family of Young measures (ν_x), $\|Dv_j - Du_j\|_{L^1} \to 0$, $f(Dv_j) \overset{*}{\rightharpoonup} \int_K f(\lambda)d\nu_x$ for all continuous functions f.

(III) For a.e. $x_0 \in \Omega$, there exists a bounded sequence (ϕ_k) in $W_0^{1,\infty}(D,\mathbb{R}^N)$ where D is the unit disk in $\mathbb{R}^n$, such that the corresponding gradient Young measures $\{\hat{\nu}_y\}$ of the sequence $(Du(x_0)+D\phi_k)$ satisfy $\hat{\nu}_y = \nu_{x_0}$ a.e. $y \in D$. We call $\hat{\nu}_y$ the homogeneous Young measure and simply denote it by $\nu := \hat{\nu}_y$, $y \in D$. The homogeneous Young measure enables us to **localize** our problem. We may decompose Theorem 1 into

Lemma 1. *('Existence') Suppose ν is a homogeneous Young measure satisfying* $\operatorname{supp}\nu \subset K_\epsilon$ *and* $\bar{\nu} = X \in \bar{B}_\epsilon(A)$. *Then* $\operatorname{supp}\nu \subset \bar{B}_\epsilon(A)$.

Lemma 1 implies that for a fixed x, $\operatorname{supp}\nu_x \subset K_\epsilon$, $\bar{\nu}_x \in \bar{B}_\epsilon(A)$ implies $\operatorname{supp}\nu_x \subset \bar{B}_\epsilon(A))$.

Lemma 2. *('Regularity') There is an estimate of $\epsilon > 0$ depending on $|A-B|$, $\lambda_{\max}$, n and N such that if $Du(x) \in K_\epsilon$ a.e. in Ω. Then either $Du(x) \in \bar{B}_\epsilon(A)$ a.e. or $Du(x) \in \bar{B}_\epsilon(B)$ a.e.*

Lemma 2 shows that if $\bar{\nu}_x \in K_\epsilon$ then $\bar{\nu}_x \in \bar{B}_\epsilon(A)$ a.e. or $\bar{\nu}_x \in \bar{B}_\epsilon(B)$. Combining (1), Lemma 1 and Lemma 2, we reach the conclusion of Theorem 1.

To prove Lemma 1 we need to control the large scale oscillation of the gradients of a sequence of mappings with a fixed affine boundary condition. For Lemma 2, we need to rule out large scale oscillation of the gradient of a fixed mapping without prescribed boundary condition. Note that in either cases the best possible we can reach is some partial rigidity of the gradients.

The tool we use is the Schauder estimates in BMO and Campanato spaces for linear elliptic systems with constant coefficients. Notice the ellipticity property of linear subspaces of $M^{N\times n}$ without rank-one matrices [2]. Let $E = \operatorname{span}[A-B]$. Then E is a subspace without rank-one matrices. Let $E^\perp$ be the orthogonal complement of E and $P_{E^\perp}$ be the orthogonal projection to $E^\perp$, then there is some constant $c_0 > 0$ such that for any rank-one matrix $a \otimes b$, $|P_{E^\perp}(a \otimes b)|^2 \geq c_0|a|^2|b|^2$, where $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$.

Let us recall some basic facts about elliptic system with constant coefficients [9]:

$$\operatorname{div} A^{ij}_{\alpha\beta} D_\beta u^j = \operatorname{div} F^i_\alpha \text{ in } \Omega \text{ with } F^i_\alpha \in L^\infty. \tag{2}$$

We say that the system satisfy the Legendre-Hadamard strong ellipticity condition if

$$A^{ij}_{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \geq \lambda_0|\xi|^2|\eta|^2,\ \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^N. \tag{3}$$

We denote by $\Lambda_0 > 0$ a constant such that $A^{ij}_{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j \leq \Lambda_0|\xi|^2|\eta|^2$.

Example. Let $E \subset M^{N\times n}$ be a subspace without rank-one matrices. Then the second order elliptic system $\operatorname{div} P_{E^\perp} Du = \operatorname{div} F$ satisfies (3). In our case $\lambda_0 = 1 - \lambda_{\max}$, and $\Lambda_0 = 1$.

Let $u \in W^{1,2}$ be a weak solution of (2). The following are some estimates of Du.

(A) **Interior Estimate**

Let $x_0 \in \Omega$ and $0 < \rho < R$ such that $B_\rho(x_0) \subset B_R(x_0) \subset \bar{B}_R(x_0) \subset \Omega$,

$$\fint_{B_\rho(x_0)} |Du - [Du]_{x_0,\rho}|^2 dx \leq C\left[\left(\frac{\rho}{R}\right)^\tau \fint_{B_R(x_0)} |Du - [Du]_{x_0,R}|^2 dx + [F]^2_{\mathcal{L}^{2,n}(\Omega)}\right], \tag{4}$$

where $C = C(n, N, \lambda_0, \Lambda_0) > 0$, $\tau = \tau(n, N, \lambda_0, \Lambda_0)$, $0 < \tau < 2$, and $[F]_{\mathcal{L}^{2,n}(\Omega)}$ is the Campanato seminorm.

(B) **Global Estimate**

Under Dirichlet condition $u|_{\partial\Omega} = 0$,

$$\|Du\|_{BMO(\Omega)} \leq C\|F\|_{L^\infty}. \tag{5}$$

We prove Lemma 2 first which depends on the interior estimate (a) above.

Proof of Lemma 2. Without loss of generality, we may assume that $A = 0$, $E = \operatorname{span}[B]$. This can be done by a simple translation in $M^{N\times n}$. Note that $|P_{E^\perp}(Du)| \leq 2\epsilon$, hence $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ is a weak solution of

$$\operatorname{div} P_{E^\perp}(Du) = \operatorname{div} F, \quad \text{where} \quad F = P_{E^\perp}(Du), \quad \|F\|_{L^\infty} \leq 2\epsilon.$$

Let $V = \{x \in \Omega,\ Du(x) \in \bar{B}_\epsilon(B)\}$. $Du(x) \in K_\epsilon$ implies that

$$Du = B\chi_V + H, \qquad \|H\|_{L^\infty} \leq 4\epsilon. \tag{6}$$

Then (a) implies, for a fixed $x_0 \in \Omega$ and $0 < \rho < R$ such that $B_{2R}(x_0) \subset \Omega$,

$$\begin{aligned}\fint_{B_\rho(x_0)} |Du - [Du]_{x_0,\rho}|^2 dx &\leq C\left[\left(\frac{\rho}{R}\right)^\tau \fint_{B_R(x_0)} |Du - [Du]_{x_0,R}|^2 dx + [F]^2_{\mathcal{L}^{2,n}(\Omega)}\right] \\ &\leq C\left(\frac{\rho}{R}\right)^\tau (|B| + 2\epsilon)^2 + C\epsilon^2 \leq C\epsilon^2\end{aligned}$$

if ρ/R is sufficiently small. Consequently, for every $x_0 \in \Omega$, there is a small cubes $Q(x_0, r) \subset \Omega$ centered at x_0 with side-length $r > 0$ depending on x_0 and Ω,

$$\|Du\|^2_{BMO(Q(x_0,r))} \le C\epsilon^2. \tag{7}$$

By applying the definition of BMO and (3), we have, for each cube $Q \subset Q(x_0, r)$ and let

$$G(Q) = \frac{|Q \cap V|}{|Q|}, \qquad \text{then} \qquad G(Q)(1 - G(Q)) \le \frac{C}{|B|^2}\epsilon^2 < \frac{3}{16},$$

as long as $\epsilon > 0$ is small. Hence for $Q \subset Q(x_0, r)$,

$$\text{either} \qquad G(Q) \le \frac{1}{4} \qquad \text{or} \qquad G(Q) \ge \frac{3}{4}. \tag{8}$$

Now we use the Intermediate Value Theorem and a density argument to finish the proof.

Without loss of generality, we assume $x_0 \in \Omega$ is a point of density 1 for V. Then we show that there is no points of density 1 for $\Omega \setminus V$ in Ω. Since x_0 is a point of density 1 for V, there is a cube $Q_0 \subset Q(x_0, r)$ centered at x_0, such that $G(Q_0) > 3/4$. We first show that there is no point of density 1 for $Q_0 \setminus V$ in Q_0. Otherwise, let $x_1 \in Q_0$ be an interior point such that there is some $Q_1 \subset Q_0$ centered at x_1 and satisfies $G(Q_1) < 1/4$. Then we may construct a continuous family of decreasing cubes $Q(t)$ in Q_0 such that $Q(0) = Q_0$ and $Q(1) = Q_1$. It is then easy to see that $t \to G(Q(t))$ is a continuous function. By the Intermediate Value Theorem, there is some cube $Q(t_0) \subset Q_0 \subset Q(x_0, r)$, such that $G(Q(t_0)) = 1/2$. This contradicts to (8).

Since Ω is arcwise connected, for each $x \in \Omega$, there is a piecewise affine curve $\gamma : [0, 1] \to \Omega$, such that $\text{dist}(\gamma, \partial\Omega) = \delta_0 > 0$, $\gamma(0) = x_0$, $\gamma(1) = x$. If we choose $r > 0$ sufficiently small and let $Q(t)$ be the cube centered at $\gamma(t)$ with radius $0 < r < \delta_0$, we may claim that (8) holds for $Q \subset Q(t)$, $0 \le t \le 1$. Then it is easy to see that x is not a point of density 1 for $\Omega \setminus V$, hence $Du(x) \in B_\epsilon(B)$ a.e. $\square$

Proof of Lemma 1. We may assume $A = 0$ as before and $\operatorname{supp} \nu \subset K_\epsilon$ and $\bar\nu = X \in \bar B_\epsilon(0)$. Let $X + Du_j$ generates ν with $u_j \in W^{1,\infty}_0$ bounded. Define

$$F_j = \begin{cases} P_{E^\perp}(Du_j), & \text{if } |P_{E^\perp}(Du_j)| \le 4\epsilon, \\ 0, & \text{otherwise,} \end{cases}$$

where $E = \text{span}[B]$. Solving $\operatorname{div} P_{E^\perp}(Dv_j) = \operatorname{div} F_j$ in D, $v_j|_{\partial D} = 0$,

$$\|Dv_j\|_{BMO(D)} \le C\epsilon, \qquad \|Du_j - Dv_j\|_{L^2} \to 0 \tag{9}$$

Let $W_j = \{x \in D, \operatorname{dist}(Dv_j, K) \geq 4\epsilon\}$ so $|W_j| \to 0$ and let $V_j = \{x \in D, |Dv_j - B| < 4\epsilon\}$, then $Dv_j = B\chi_{V_j} + Dv_j\chi_{W_j} + O(\epsilon)$. Let $G_j(Q) = |V_j \cap Q|/|Q|$, then (9) implies

$$G_j(Q)(1 - G_j(Q)) \leq C\epsilon^2 + C \fint_Q (1 + |Dv_j|^2)\chi_{W_j} dx. \tag{10}$$

Also notice that $\int_D Dv_j dx = 0$ which implies $G_j(D) \leq C\epsilon < 1/4$ for large j.

Our aim is to show that $|V_j| \to 0$ so that $\operatorname{dist}^2(X + Dv_j, \bar{B}_\epsilon) \to 0$ in L^1, hence $\nu \operatorname{supp} \bar{B}_\epsilon(0)$. Now we use a slightly different argument as in the proof of Lemma 1 by using $\int_{W_j}(1 + |Dv_j|^2)dy$ to bound $|V_j|$.

For each point $x \in D$ of density 1 for V_j, there exists a cube $Q \subset D$ centered at x such that $G_j(Q) > 3/4$. Note that $G_j(D) < 1/4$, we can then prove that there is an open cube Q_x containing Q in D such that $G_j(Q_x) = 1/2$ which maximize the left hand side of (10). If we further require that $1/4 - C\epsilon^2 := \gamma > 0$, then from (10),

$$\gamma|Q_x| = |Q_x|\left(\frac{1}{4} - C\epsilon^2\right) \leq C \int_{Q_x} (1 + |Dv_j|^2)\chi_{W_j} dy. \tag{11}$$

Clearly $\{Q_x\}$ is a covering of the points of density 1 for V_j by open cubes. By Besicovitch's covering lemma (see e.g. [8]), it is then easy to prove that

$$\gamma|V_j| \leq C \int_{W_j} (1 + |Dv_j|^2)dy \to 0.$$

Therefore $|V_j| \to 0$ so that $\operatorname{supp} \nu \subset \bar{B}_\epsilon(0)$. □

Theorem 1 can be generalized to any finite sets contained in a subspace without rank-one matrices [19]. For a finite set $K = \{A_i\} \subset M^{N\times n}$, we define the diameter of K $d_K = \max\{|A_i - A_j|, i \neq j\}$, and the smallest distance $g_K = \min\{|A_i - A_j|, i \neq j\}$.

Theorem 2. *Suppose $E \subset M^{N\times n}$ be a linear subspace without rank-one matrices. Let $K = \{A_i\} \subset E$ be a finite subset. Let*

$$\lambda_E = \min\{|P_{E^\perp}(a \otimes b)|^2,\ a \in \mathbb{R}^N,\ b \in \mathbb{R}^n,\ |a| = |b| = 1\},$$

$$\frac{1}{\mu_E} = \inf_{|a|=|b|=1} \frac{|P_E(a \otimes b)|^2}{|P_{E^\perp}(a \otimes b)|^2} = \frac{1 - \lambda_E}{\lambda_E}.$$

Then there exists an estimate of $\epsilon > 0$ depending on d_K, g_K, λ_E, μ_E, n and N, such that $u_j \rightharpoonup u$ in $W^{1,1}$, $\int_\Omega \operatorname{dist}(Du_j, K_\epsilon) \to 0$ implies, up to a subsequence, for some fixed $A_{i_0} \in K$,

$$\lim_{j\to\infty} \int_\Omega \operatorname{dist}(Du_j, \bar{B}_\epsilon(A_{i_0})) \to 0.$$

Remark 1. Theorem 2 was proved in [19] by using S. Müller's improved approximation lemma for sequences of gradients approximating a compact set [14]. In the special case of Theorem 1, we may avoid using this result, instead, establishing Lemma 1 directly form the global estimate for the Dirichlet problem.

Remark 2. For the incompatible multi-elastic well structure $K = \bigcup_{i=1}^{m} SO(2)H_i$ with H_i positive definite, the theory for linear elliptic system still works provided that the wells are sufficiently 'flat' [20].

Remark 3. For the two well structure $K = SO(n) \cup SO(n)H$, it is known [12, 16] that under a technical assumption on H, the compactness result holds, that is, if $\mathrm{dist}^2(Du_j, K) \to 0$ in L^1, then there exists some $A \in K$, such that $Du_j \to A$ a.e. A nonlinear elliptic system is involved. However, I do not know any interior BMO estimates for the elliptic system $\operatorname{div} A(Du) = \operatorname{div} f$, $u|_{\partial\Omega} = 0$, where $c|Y|^2 \le DA(X)YY \le C|Y|^2$, $\|f\|_{L^\infty} \le \epsilon$?

As remarked in [12], if $H = \lambda I$ with $\lambda > 0$, $\lambda \neq 1$, and I being the identity matrix, one may simply use the n-Laplace operator to study convergent sequences of gradients to K, that is, $\operatorname{div} |Du|^{n-2}Du = \operatorname{div} F$. However, even for this explicit system, I do not know any BMO estimate of the weak solutions in $W^{1,n}$ given that $\|F\|_{L^\infty}$ is small.

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